

6.

CTNT 2024

Connecticut Summer School in Number Theory

Class Field Theory

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Abstract CFT ingredients

- G a profinite group $G = \lim_{\leftarrow} H$
 $\text{Gal}(\bar{k}/k)$ finite
- A a continuous multiplicative G -module
- $\text{id}(a) = a$ $A = \bar{k}^\times$
- $\sigma(ab) = \sigma(a)\sigma(b)$ $= \bar{\mathbb{Q}_p}^\times$
- $(\sigma \circ \tau)(a) = \tau(\sigma(a))$
- $A = \bigcup_{H \leq G} A^H$ $A^H = \{a \in A : \sigma(a) = a \text{ } \forall \sigma \in H\}$
 $\text{open } \} H = \text{Gal}(\bar{k}/k) [K:k] < \infty$

- $d: G \rightarrow \hat{\mathbb{Z}}$ continuous

- $v: A \rightarrow \hat{\mathbb{Z}}$

$$v(A) \geq \mathbb{Z} \quad v(A)/nV(A) \cong \mathbb{Z}/n\mathbb{Z}$$

$$v(N_{K/k} A_K) = f_K v(A) \quad K/k \text{ finite}$$

\sqcup

$$n^H \quad H = \text{Gal}(\bar{k}/k)$$

Class field theory axioms

For every finite cyclic extension L/K
 $L \supseteq K \supseteq k$ $\text{Gal}(L/k) = \langle \sigma \rangle$

$$\frac{\#\mathcal{A}_K}{N_{L/K}(\mathcal{A}_L)} = [L:K]$$

. if $a \in \mathcal{A}_L$ of norm 1 then $a = \frac{\sigma(b)}{b}$
 $b \in \mathcal{A}_L$

Kummer theory

Axiom: For every finite cyclic extension

L/K if

$a \in A_L$ with norm 1 then $a = \frac{\sigma(b)}{b}$ $b \in A_L$

Today: $G = \text{Gal}(\bar{k}/k)$ \bar{k} field

$A = \bar{k}^\times$ $\sim k$ any field

In this case axiom is always satisfied
this is Hilbert 90

Fix n positive integer, assume

$\mu_n \subseteq K$ $\mu_n = \text{set of } n^{\text{th}} \text{ roots of unity}$
 K/\mathbb{K}

Definition: A Kummer extension of K is of the form

$$K(\sqrt[n]{\Delta})$$

$$\Delta \subseteq K^\times$$

all n^{th} roots of all elements
in Δ

Definition: A Kummer extension of K
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$$K(\sqrt[n]{\Delta})$$

$$\Delta \subseteq K^{\times}$$

Such an extension is Galois with abelian
Galois group of exponent n



$$\forall g \in \text{Gal}(K(\sqrt[n]{\Delta})/K)$$

$$g^n = \text{id}$$

n least positive such integer

Such an extension is Galois with
Galois group of exponent n

To see this: $a \in \Delta$

$$\begin{aligned} \text{Gal}(\mathbb{K}(\sqrt[n]{a})/\mathbb{K}) &\rightarrow \mu_n \\ \sigma &\mapsto \frac{\sigma(\sqrt[n]{a})}{\sqrt[n]{a}} \end{aligned}$$

"Classification" theorem

If L/K is abelian of exponent n

then

$$L = K(\sqrt[n]{\Delta}) \quad \text{with} \quad \Delta = (L^\times)^n \cap K^\times$$

PROOF for L/K finite cyclic

In this case, $L = K(\alpha)$ with $\alpha^n \in K^\times$

Let $\text{Gal}(L/K) = \langle \sigma \rangle$, $\langle \tau \rangle = \mu_n$

$\tau \in K$ so $N_{L/K}(\tau) = \tau^n = 1$

By the axiom / Hilbert 90

$\tau = \frac{\sigma(\alpha)}{\alpha}$ for some $\alpha \in L^\times$

So $K \subseteq K(\alpha) \subseteq L$

$\sigma^i(\alpha) = \tau^i \alpha$ so $\sigma^i(\alpha) = \alpha$ iff
 $i \equiv 0 \pmod{n}$

So $[K(\alpha) : K] = n$ but so does $[L : K]$ and so $L = K(\alpha)$,

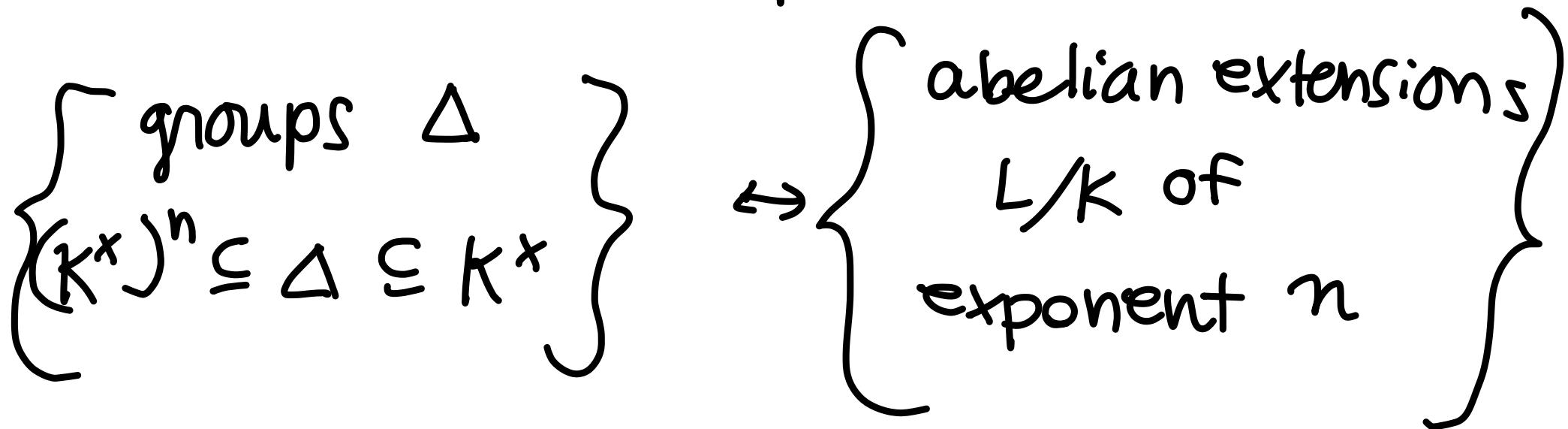
Now $\frac{\sigma(\alpha^n)}{\alpha^n} = \left(\frac{\sigma(\alpha)}{\alpha} \right)^n = \zeta^n = 1$

so $\sigma(\alpha^n) = \alpha^n$ and $\alpha^n \in K$



As a consequence if $\gcd(\text{char } K, n) = 1$ $\mu_n \subseteq K^\times$

There is a 1-1 correspondence



under this correspondence

$$\Delta = (L^\times)^n \cap K^\times \quad \text{and}$$

$$\begin{cases} \text{Gal}(L/K) \cong \text{Hom}(\Delta / (K^\times)^n, \mu_n) \\ \text{Hom}(\text{Gal}(L/K), \mu_n) \cong \Delta / (K^\times)^n \end{cases}$$

Hypotheses from now on

$\gcd(\text{char } K, n) = 1$ $\mu_n \subseteq K^\times$, K local field

(K/\mathbb{Q}_p finite, $K = \mathbb{R}, \mathbb{C}$)

$L = K(\sqrt[n]{K^\times})$ maximal abelian extension
of K of exponent n

Fact (should be proved)

$$N_{L/K}(L^\times) = (K^\times)^n$$

Reciprocity
law

By CFT

$$\text{Gal}(L/K) \cong K^\times / (K^\times)^n$$

CFT

$$\text{Gal}(L/K) \cong K^\times / (K^\times)^n$$

Kummer theory

$$\text{Hom}(\text{Gal}(L/K), \mu_n) \cong K^\times / (K^\times)^n$$

We get the Hilbert symbol

$$(\frac{-}{\sigma}) : K^\times / (K^\times)^n \times K^\times / (K^\times)^n \rightarrow \mu_n$$

prime of K $\xrightarrow{\quad}$

$$(\underset{\uparrow}{\sigma}, \underset{\uparrow}{x}) \mapsto x(\sigma)$$

$$\text{Gal}(L/K) \quad \text{Hom}(\text{Gal}(L/K), \mu_n)$$

$$a, b \in K^\times$$

$$r_{K(\sqrt[n]{b})/K} : \text{Gal}(K(\sqrt[n]{b})/K) \xrightarrow{\sim} K^\times / N_{K(\sqrt[n]{b})/K}(K(\sqrt[n]{b})^\times)$$

$$\sigma_a \longleftrightarrow a$$

then

$$\left(\frac{a, b}{\phi} \right) = \frac{\sigma_a(\sqrt[n]{b})}{\sqrt[n]{b}}$$

$$n=2 \quad \mu_2 = \{\pm 1\} \subseteq \mathbb{Q}_p$$

$$\left(\frac{p, a}{p}\right) = \left(\frac{a}{p}\right) \quad \text{Legendre symbol}$$

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Nun einem bei
G der Or
gehörige

tzung) zu
ment aus
der zu g^σ

Ist $\psi_i(\tau)$ ein Charakter (also gewöhnlicher Abel'scher Gruppencharakter) von $g^\sigma(i = 1, 2 \dots m(\sigma))$, wo $i = 1$ der Hauptcharakter sei), so bilden wir in Ω die L -Reihe in bezug auf K :

$$L(s, \psi_i^{(\sigma)}):$$

Wegen (15) und (7) gilt, wenn wir noch den Index σ zur Unterscheidung anbringen:

...That's all for now ...

$$(23) \quad L(s, \psi_i^{(\sigma)}) = \prod_{\nu=1}^r (L(s, \chi_\nu))^{r_{i\nu}^{(\sigma)}} \quad (i = 1, 2, \dots m(\sigma)).$$

Nach der gemachten Voraussetzung steht links eine gewöhnliche Abel'sche L -Reihe. Deshalb gestattet (23) die Fortsetzbarkeit unserer Funktionen zu beweisen. Zunächst ist $L_i(s, \chi^1) = f_i(s)$. Für sie ist also die Fortsetzbarkeit bewiesen.