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Connecticut Summer School in Number Theory

Class Field Theory

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Abstract CFT ingredients

- G a profinite group $\quad G = \varprojlim_{\text{finite}} H$

  $= \mathrm{Gal}(\bar{k}/k)$

- A a continuous multiplicative G-module $\quad A = \bar{k}^\times = \bar{\mathbb{Q}}_p^\times$
  - $\mathrm{id}(a) = a$
  - $\sigma(ab) = \sigma(a)\sigma(b)$
  - $(\sigma \circ \tau)(a) = \tau(\sigma(a))$
  - $A = \bigcup_{\substack{H \leq G \\ \text{open}}} A^H \quad A^H = \{a \in A : \sigma(a) = a \ \forall \sigma \in H\}$

    $\{ H = \mathrm{Gal}(\bar{k}/K) \quad [K:k] < \infty$

- $d: G \twoheadrightarrow \hat{\mathbb{Z}}$ continuous
- $v: A \to \hat{\mathbb{Z}}$

$$v(A) \supseteq \mathbb{Z} \qquad v(A)/nv(A) \cong \mathbb{Z}/n\mathbb{Z}$$

$$v(N_{K/k} \underbrace{A_K}_{A^H}) = f_K\, v(A) \qquad K/k \text{ finite}$$

$H = \mathrm{Gal}(\bar{k}/K)$

Class field theory axioms

For every finite cyclic extension L/k

$L \supseteq K \supseteq k$ $\qquad$ $Gal(L/K) = \langle \sigma \rangle$

- $\# A_K / N_{L/K}(A_L) = [L:K]$

- if $a \in A_L$ of norm 1 then $a = \dfrac{\sigma(b)}{b}$, $b \in A_L$

Kummer theory

Axiom: For every finite cyclic extension L/K if $a \in A_L$ with norm 1 then $a = \frac{\sigma(b)}{b}$ $b \in A_L$

Today: $G = Gal(\bar{k}/k)$ k field
$A = \bar{k}^{\times}$

In this case axiom is always satisfied ~ k any field
this is Hilbert 90

Fix n positive integer, assume

$\mu_n \subseteq K$ $\quad$ μ_n = set of n^{th} roots of unity

K/k

Definition: A Kummer extension of K is of the form

$K(\underbrace{\sqrt[n]{\Delta}}_{\text{all nth roots of all elements in } \Delta})$ $\quad$ $\Delta \subseteq K^\times$

Definition: ~~A~~ Kummer extension of K is of the form

$$K(\sqrt[n]{\Delta}) \qquad \Delta \subseteq K^{\times}$$

Such an extension is Galois with abelian Galois group of exponent n

$\forall g \in Gal(K(\sqrt[n]{\Delta})/K)$

$g^n = id$

n least positive such integer

Such an extension is Galois with Galois group of exponent n

To see this: $a \in \Delta$

$$\mathrm{Gal}(K(\sqrt[n]{a})/K) \to \mu_n$$
$$\sigma \mapsto \frac{\sigma(\sqrt[n]{a})}{\sqrt[n]{a}}$$

"Classification" theorem

If L/K is abelian of exponent n

then $L = K(\sqrt[n]{\Delta})$ with $\Delta = (L^\times)^n \cap K^\times$

proof for L/K finite cyclic

In this case, $L = K(\alpha)$ with $\alpha^n \in K^\times$

Let $\mathrm{Gal}(L/K) = \langle \sigma \rangle$, $\langle \zeta \rangle = \mu_n$

$\zeta \in K$ so $N_{L/K}(\zeta) = \zeta^n = 1$

By the axiom / Hilbert 90

$\zeta = \dfrac{\sigma(\alpha)}{\alpha}$ for some $\alpha \in L^\times$

So $K \subseteq K(\alpha) \subseteq L$

$\sigma^i(\alpha) = \zeta^i \alpha$ so $\sigma^i(\alpha) = \alpha$ iff $i \equiv 0 \bmod n$

So $[K(\alpha):K] = n$ but so does $[L:K]$ and so $L = K(\alpha)$.

Now $\dfrac{\sigma(\alpha^n)}{\alpha^n} = \left(\dfrac{\sigma(\alpha)}{\alpha}\right)^n = \zeta^n = 1$

so $\sigma(\alpha^n) = \alpha^n$ and $\alpha^n \in K$

$\square$

As a consequence if $\gcd(\operatorname{char} K, n) = 1$ $\mu_n \subseteq K^\times$

There is a 1-1 correspondence

$$\left\{ \begin{array}{c} \text{groups } \Delta \\ (K^\times)^n \subseteq \Delta \subseteq K^\times \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{abelian extensions} \\ L/K \text{ of} \\ \text{exponent } n \end{array} \right\}$$

under this correspondence

$$\Delta = (L^\times)^n \cap K^\times \quad \text{and}$$

$$\begin{cases} \operatorname{Gal}(L/K) \cong \operatorname{Hom}(\Delta/(K^\times)^n, \mu_n) \\ \operatorname{Hom}(\operatorname{Gal}(L/K), \mu_n) \cong \Delta/(K^\times)^n \end{cases}$$

Hypotheses from now on

$\gcd(\operatorname{char} K, n) = 1$ $\mu_n \subseteq K^\times$, K local field

($K/\mathbb{Q}_p$ finite, $K = \mathbb{R}, \mathbb{C}$)

$L = K(\sqrt[n]{K^\times})$ maximal abelian extension of K of exponent n

Fact (should be proved)

$$N_{L/K}(L^\times) = (K^\times)^n$$

Reciprocity law

By CFT $\quad \operatorname{Gal}(L/K) \cong K^\times/(K^\times)^n$

CFT

$$\mathrm{Gal}(L/K) \cong K^\times/(K^\times)^n$$

Kummer theory

$$\mathrm{Hom}(\mathrm{Gal}(L/K), \mu_n) \cong K^\times/(K^\times)^n$$

we get the Hilbert symbol

$$\left(\frac{-,-}{\mathfrak{p}}\right) : K^\times/(K^\times)^n \times K^\times/(K^\times)^n \to \mu_n$$

prime of K

$$(\underset{\substack{\Uparrow \\ \mathrm{Gal}(L/K)}}{\sigma}, \underset{\substack{\Uparrow \\ \mathrm{Hom}(\mathrm{Gal}(L/K),\mu_n)}}{\chi}) \mapsto \chi(\sigma)$$

$a, b \in K^\times$

$$r_{K(\sqrt[n]{b})/K} : \mathrm{Gal}(K(\sqrt[n]{b})/K) \xrightarrow{\sim} K^\times / N_{K(\sqrt[n]{b})/K}(K(\sqrt[n]{b})^\times)$$

$$\sigma_a \longleftarrow\!\shortmid \; a$$

then

$$\left(\frac{a, b}{\mathfrak{p}}\right) = \frac{\sigma_a(\sqrt[n]{b})}{\sqrt[n]{b}}$$

$n=2 \quad \mu_2 = \{\pm 1\} \subseteq \mathbb{Q}_p$

$$\left(\frac{p, a}{p}\right) = \left(\frac{a}{p}\right)$$ Legendre symbol

... That's all for now ...