CTNT 2024

Connecticut Summer School in Number Theory

Introduction to Elliptic Curves

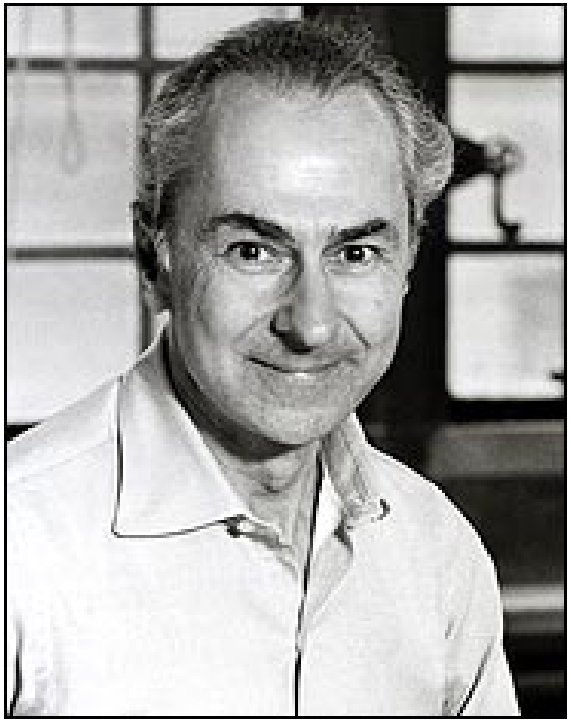
Álvaro Lozano-Robledo
(University of Connecticut)

Lecture 1

What is an elliptic curve?

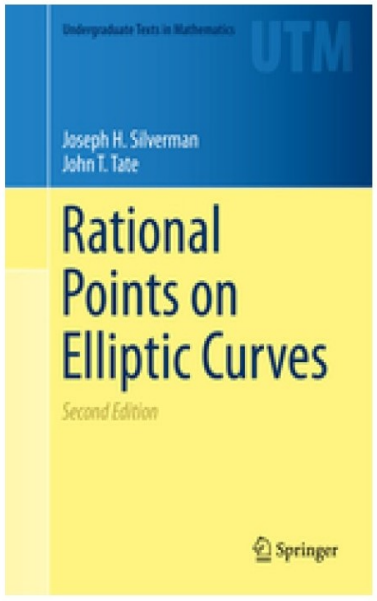
"It is possible to write endlessly on elliptic curves. (This is not a threat.)"

– Serge Lang, from *Elliptic Curves: Diophantine Analysis*



Foreword

It is possible to write endlessly on elliptic curves. (This is not a threat.) We deal here with diophantine problems, and we lay the foundations, especially for the theory of integral points. We review briefly the analytic theory of the Weierstrass function, and then deal with the arithmetic aspects of the addition formula, over complete fields and over number fields, giving rise to the theory of the height and its quadraticity. We apply this to integral points, covering the inequalities of diophantine approximation both on the multiplicative group and on the elliptic curve directly. Thus the book splits naturally in two parts.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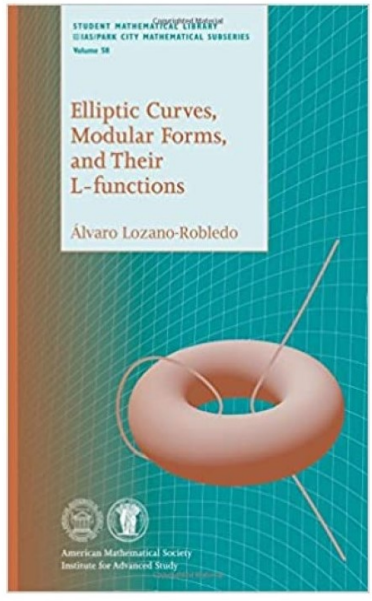
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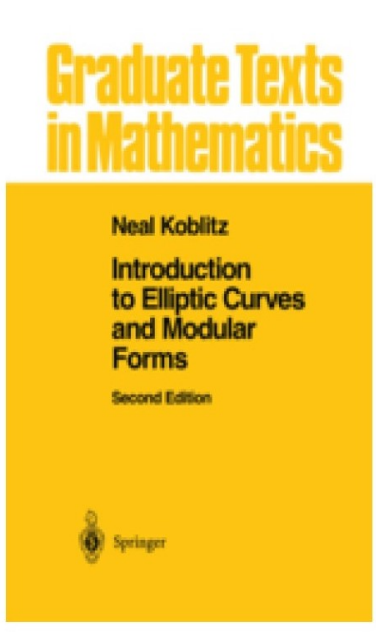
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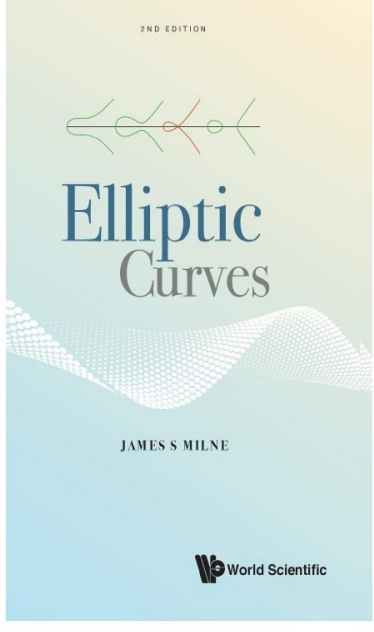
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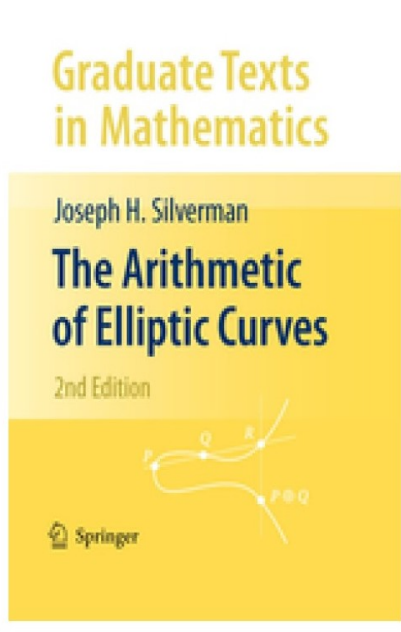
World Scientific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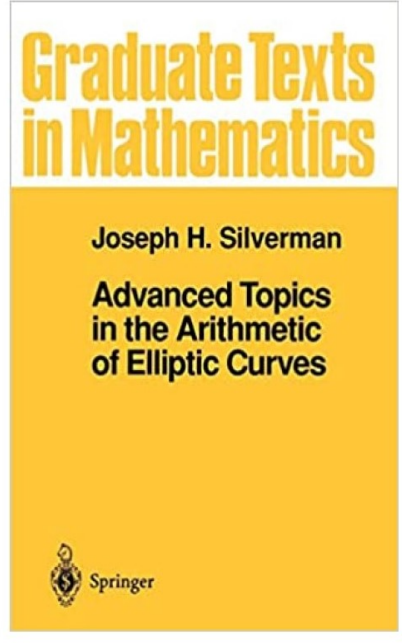
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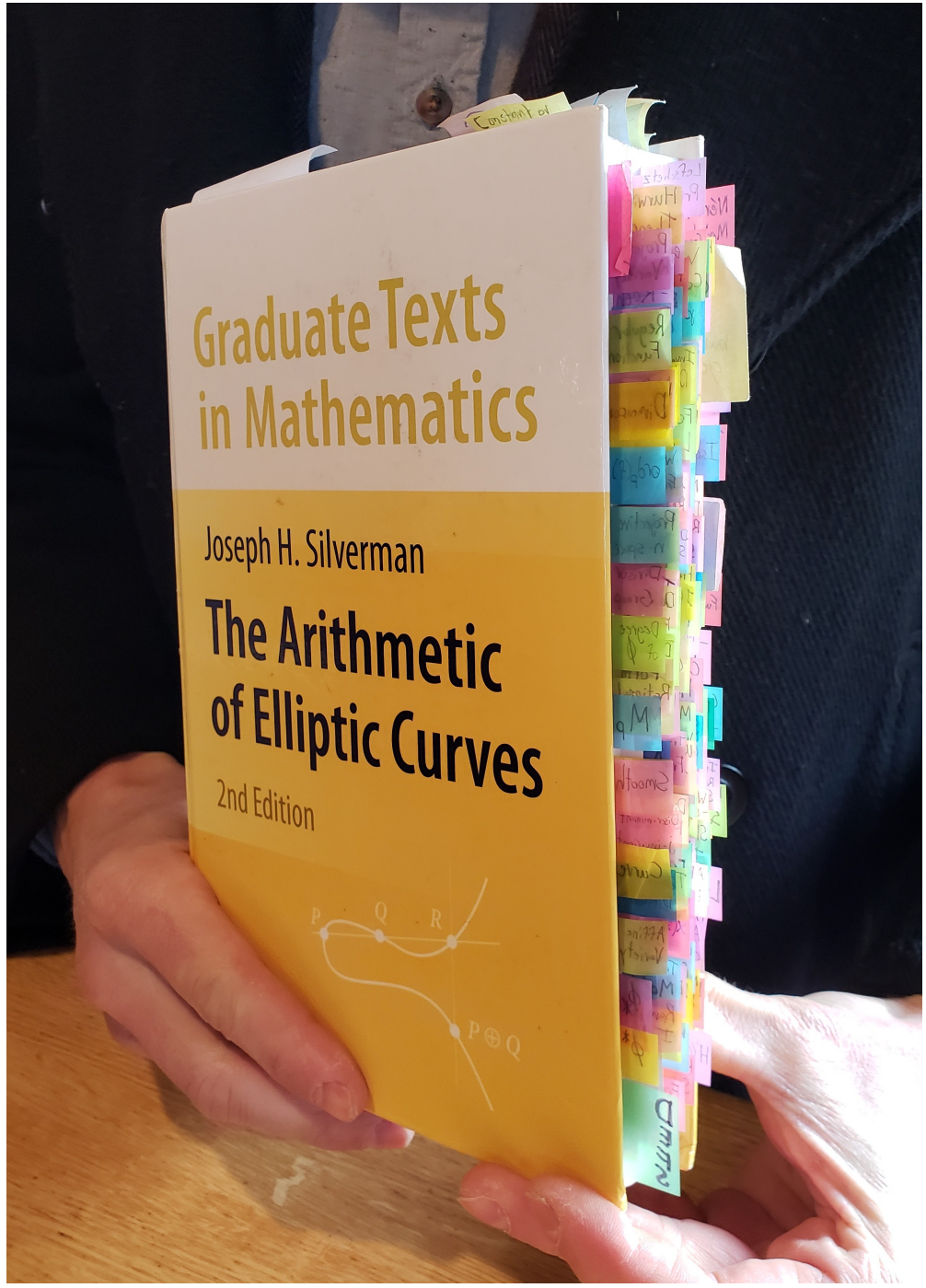
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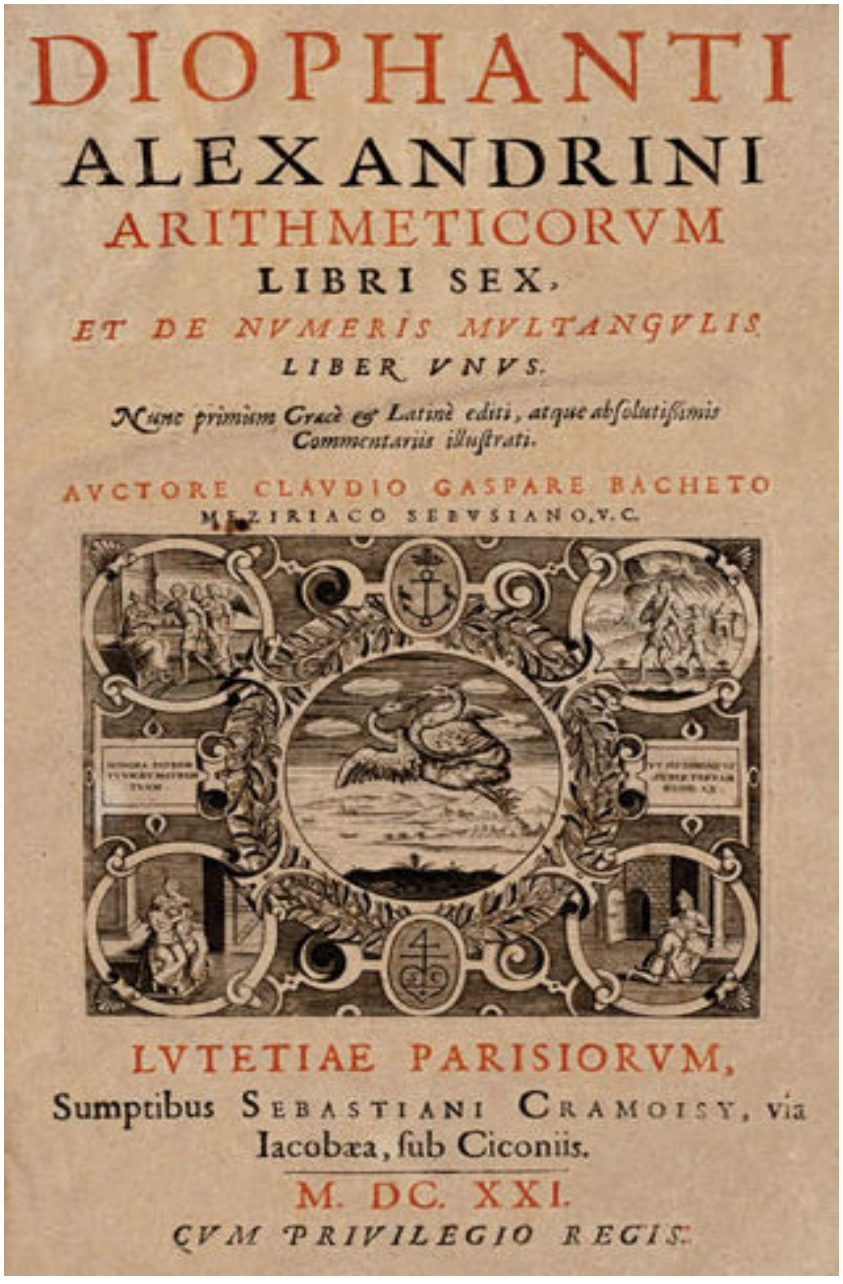

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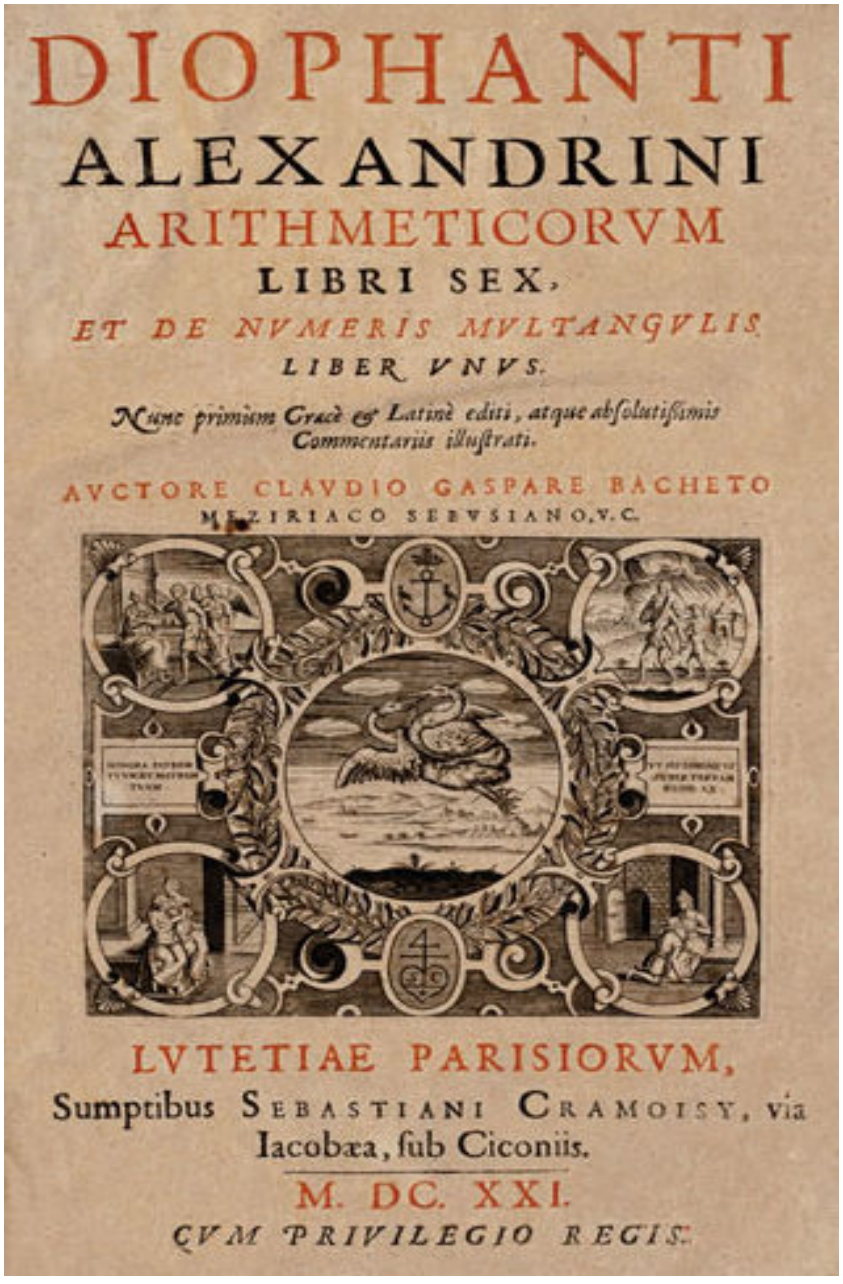
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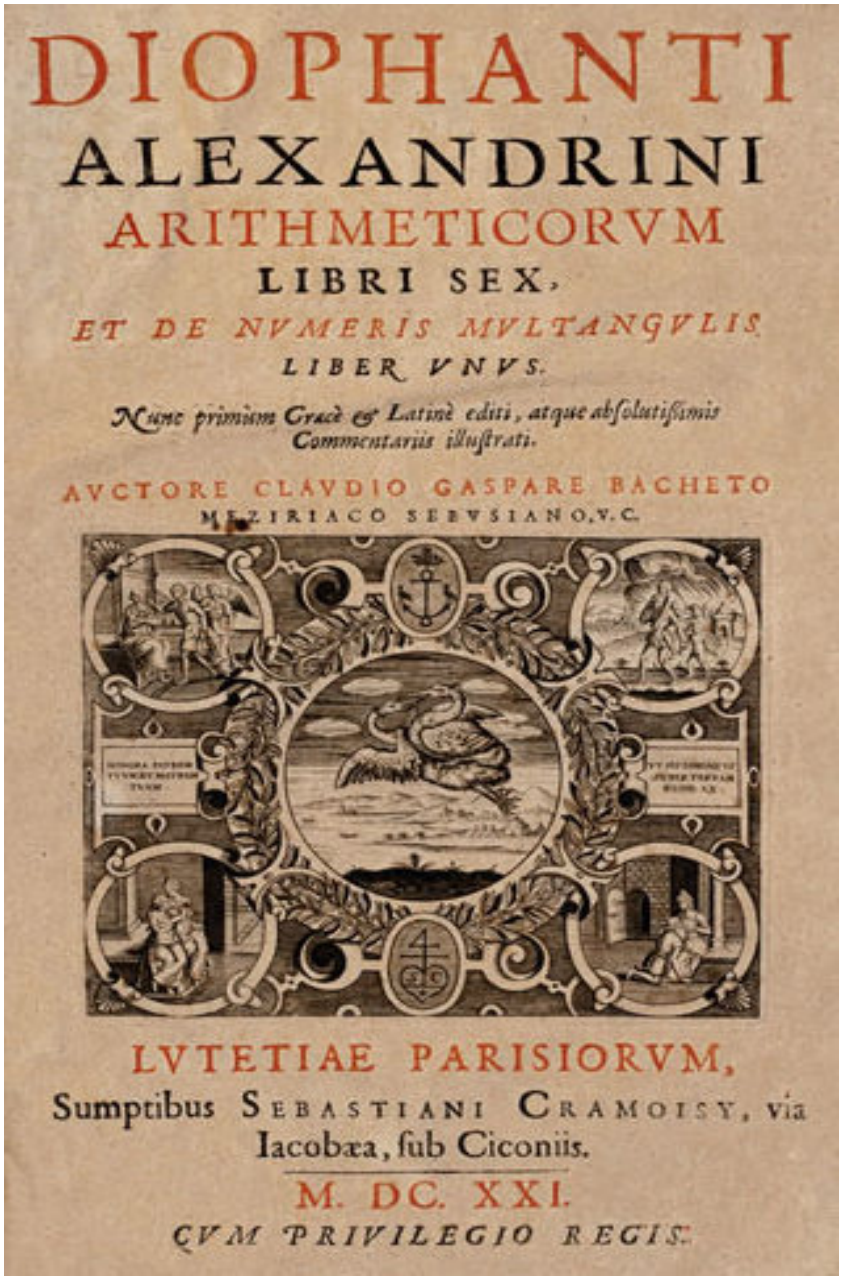

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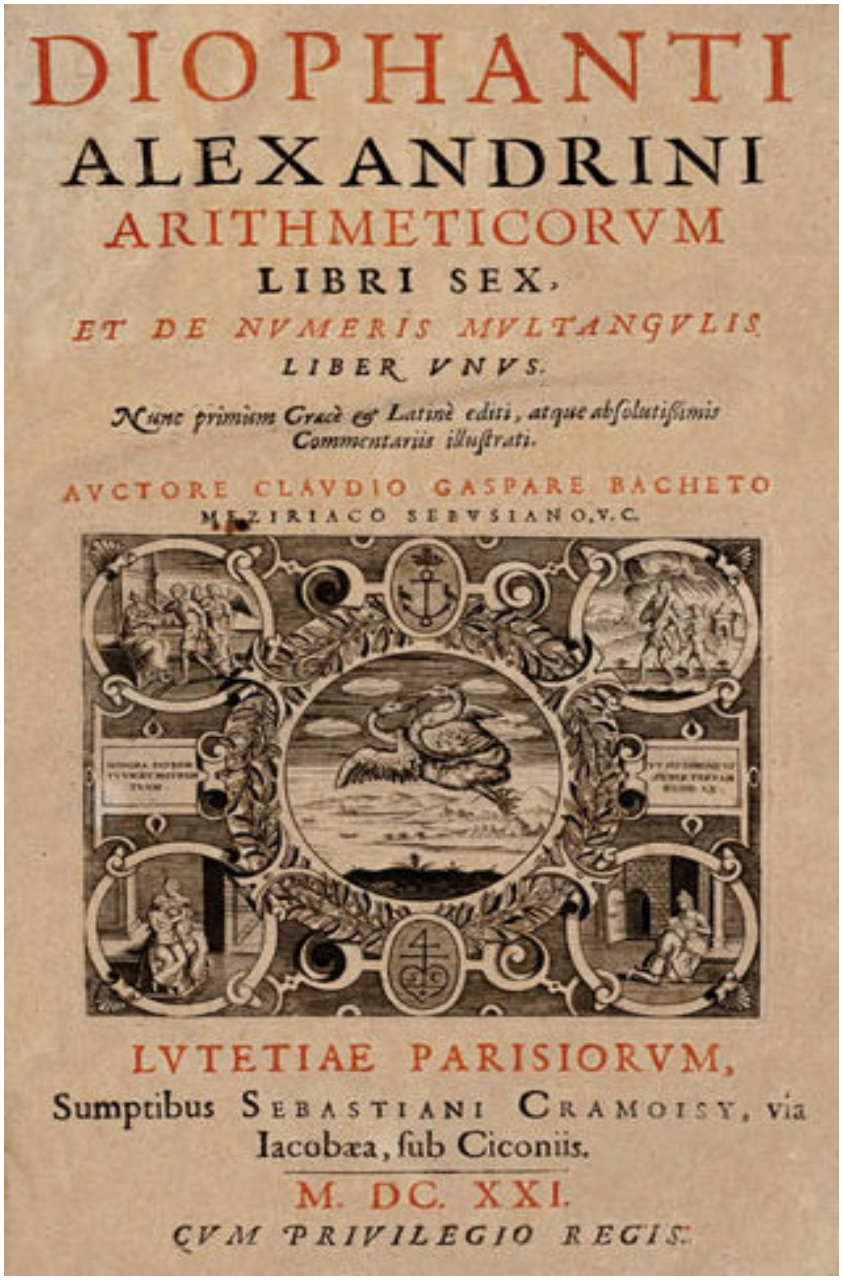
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  $(1, 1, 2)$ is a (CM) point on the cursed curve.

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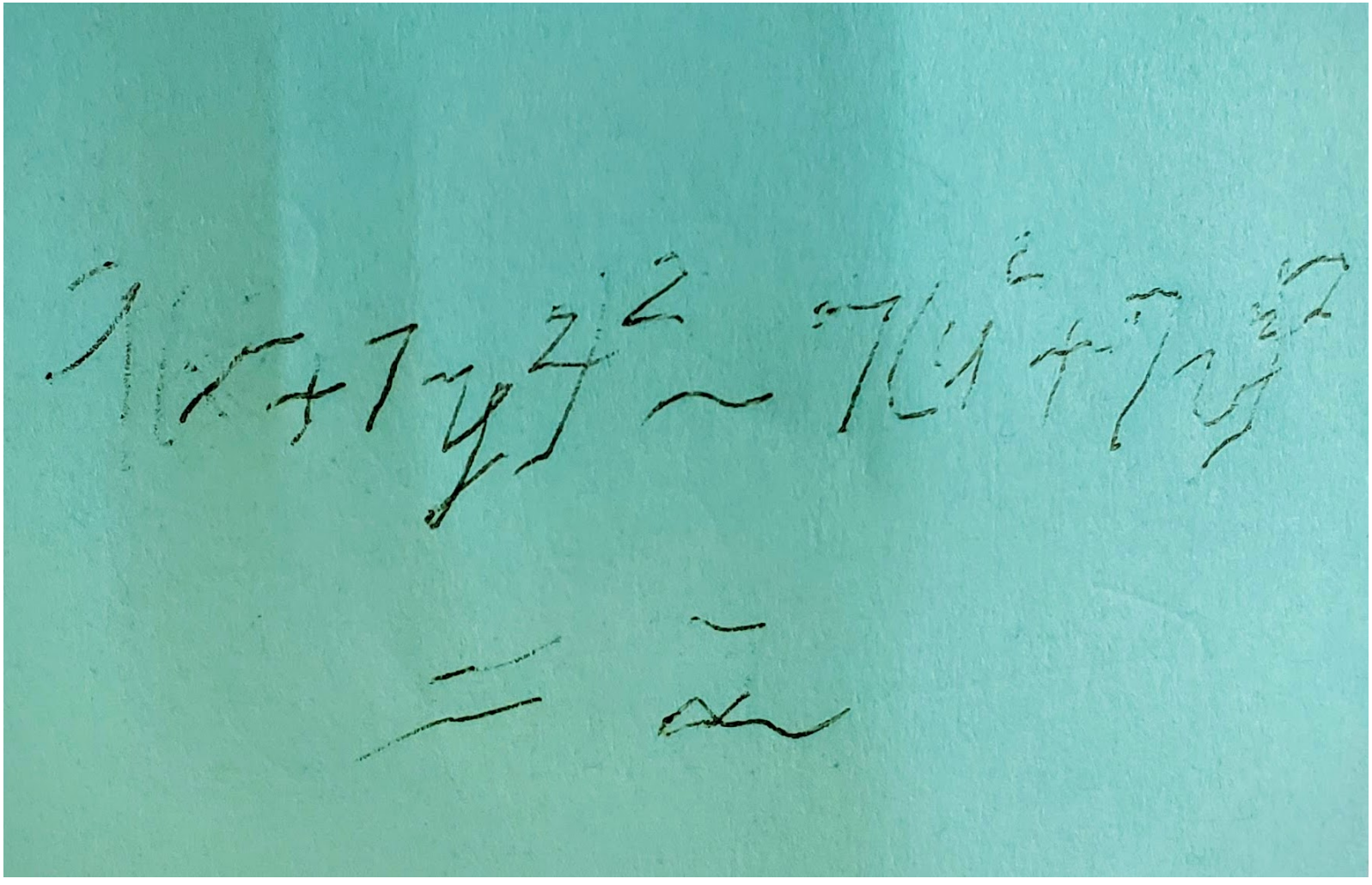
https://doi.org/10.4007/annals.2019.189.3.6

Explicit Chabauty–Kim for the split Cartan modular curve of level 13

By Jennifer S. Balakrishnan, Netan Dogra, J. Steffen Müller, Jan Tuitman, and Jan Vonk

Abstract

We extend the explicit quadratic Chabauty methods developed in previous work by the first two authors to the case of non-hyperelliptic curves. This results in a method to compute a finite set of p-adic points, containing the rational points, on a curve of genus $g \geq 2$ over the rationals whose Jacobian has Mordell–Weil rank g and Picard number greater than one, and which satisfies some additional conditions. This is then applied to determine the rational points of the modular curve $X_{\mathrm{s}}(13)$, completing the classification of non-CM elliptic curves over $\mathbf{Q}$ with split Cartan level structure due to Bilu–Parent and Bilu–Parent–Rebolledo.



A gift from Martin Davis, the diophantine equation

$$9(x^2 + 7y^2)^2 - 7(u^2 + 7v^2)^2 = 2.$$

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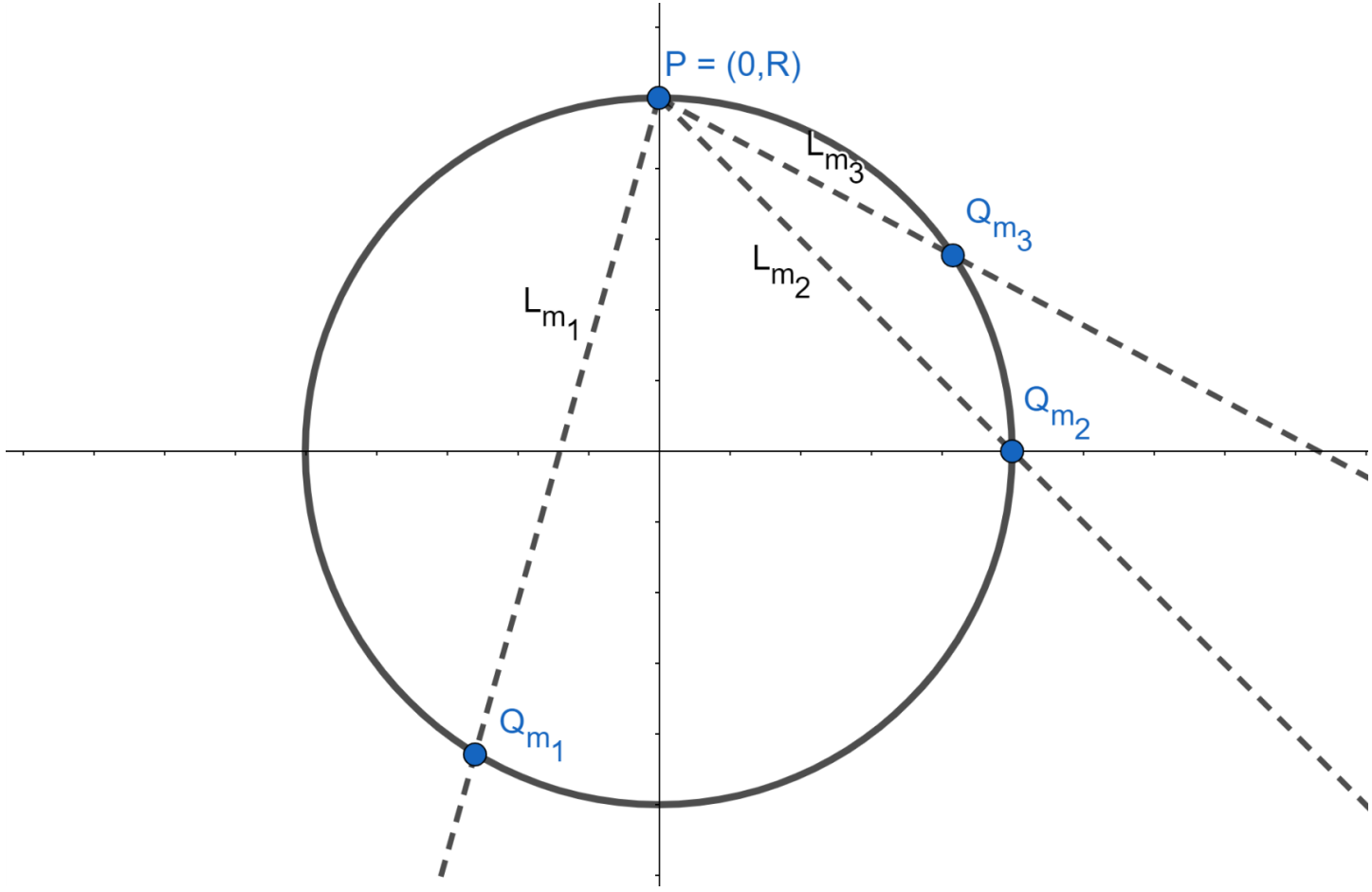
  Hasse–Minkowski (local-to-global) theory determines existence of one point. Stereographic projection finds the rest.



A parametrization via stereographic projection of the rational points on the circle $x^2 + y^2 = R^2$ of radius R is given by

$$Q_m = \left(-\frac{2Rm}{(m^2+1)}, \frac{R(m^2-1)}{(m^2+1)} \right).$$

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When C is smooth (projectively!), of degree 3 (genus 1), we lack an algorithm that will determine whether there are **any** rational points on C, or, if one exists, an algorithm that will determine **all** the rational points on the curve C.

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- **Fact:** every elliptic curve has a (Weierstrass) model of the form

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, \text{ for some } a_i \in F.$$

- We are interested in determining all F-rational points on E:

$$E(F) = \{(x_0, y_0) \in E : x_0, y_0 \in F\} \cup \{\mathcal{O} = [0 : 1 : 0]\}.$$

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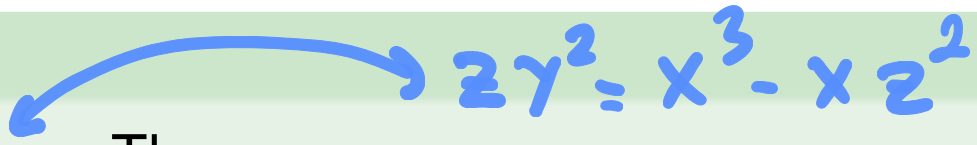

Let $E/\mathbb{Q}$ be the curve $y^2 = x^3 - x$. Then:

$$E(\mathbb{Q}) = \{\mathcal{O}, (0,0), (1,0), (-1,0)\},$$

where $\mathcal{O} = [0 : 1 : 0]$, in projective coordinates, in the "point at infinity."

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Let $E/\mathbb{Q}$ be the curve $X^3 + Y^3 = 1$.

$x^3+y^3=1729$

$\rightarrow x^3+y^3=z^3$

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Let $E/\mathbb{Q}$ be the curve $X^3 + Y^3 = 1$. Then, $E(\mathbb{Q})$ is in bijection with $E'(\mathbb{Q})$, where $E' : y^2 = x^3 - 432$ via $\psi\colon E \to E'$ given by

$$\psi((X, Y)) = \left(\frac{12}{X+Y}, \frac{36(X-Y)}{X+Y}\right), \ \psi^{-1}((x, y)) = \left(\frac{36+y}{6x}, \frac{36-y}{6x}\right).$$

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- **Hilbert's Tenth Problem** over a ring of integers of a number field F can be shown to be undecidable if a well-known conjecture (finiteness of Sha) holds for elliptic curves over F.
- **Elliptic curve cryptography** is widely used in internet applications (e.g., WhatsApp end-to-end encryption).

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The right triangles are parametrized $(e^2 - f^2, 2ef, e^2 + f^2)$ for $e > f \geq 1$. Hence, $n = ef(e^2 - f^2)$ is a congruent number.

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The right triangles are parametrized $(e^2 - f^2, 2ef, e^2 + f^2)$ for $e > f \geq 1$. Hence, $n = ef(e^2 - f^2)$ is a congruent number. For instance, $n = 30$ is the area of $(5, 12, 13)$.

The Congruent Number Problem

Let $n \geq 1$ be a natural number. Is there a right triangle (a, b, c) with rational sides $a, b, c \in \mathbb{Q}$ whose area is precisely n?

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The number $n = 1$ is *not* the area of a right triangle with rational sides (proved by Fermat). The number $n = 5$ is a congruent number, but it is not the area of a right triangle with *integer* side lengths.

In *Flos* (circa 1225), **Leonardo “Bigollo” Pisano**

In *Flos* (circa 1225), **Leonardo "Bigollo" Pisano (a.k.a. Fibonacci)** found a right triangle of area $n = 5$ in response to a challenge by the Roman Emperor Frederick II:

$$\left(\frac{3}{2}, \frac{20}{3}, \frac{41}{6}\right).$$


LEONARDO FIBONACCI
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INSIGNE MATEMATICO
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SECOLO XII

The connection between congruent numbers and elliptic curves:

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Theorem (Congruent numbers $\leftrightarrow$ Points on elliptic curves)

There is a 1-1 correspondence between the sets

- $\{(a,b,c) : a^2 + b^2 = c^2,\ \frac{ab}{2} = n\}$ *and*
- $\{(x,y) : y^2 = x^3 - n^2 x,\ y \neq 0\}$,

given by

$$(a,b,c) \mapsto \left(\frac{nb}{c-a}, \frac{2n^2}{c-a}\right),\ (x,y) \mapsto \left(\frac{x^2-n^2}{y}, \frac{2nx}{y}, \frac{x^2+n^2}{y}\right).$$

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Fibonacci's triangle $\left(\frac{3}{2},\frac{20}{3},\frac{41}{6}\right)$ of area $n=5$ maps to the point

$$P=\left(\frac{25}{4},\frac{75}{8}\right)$$

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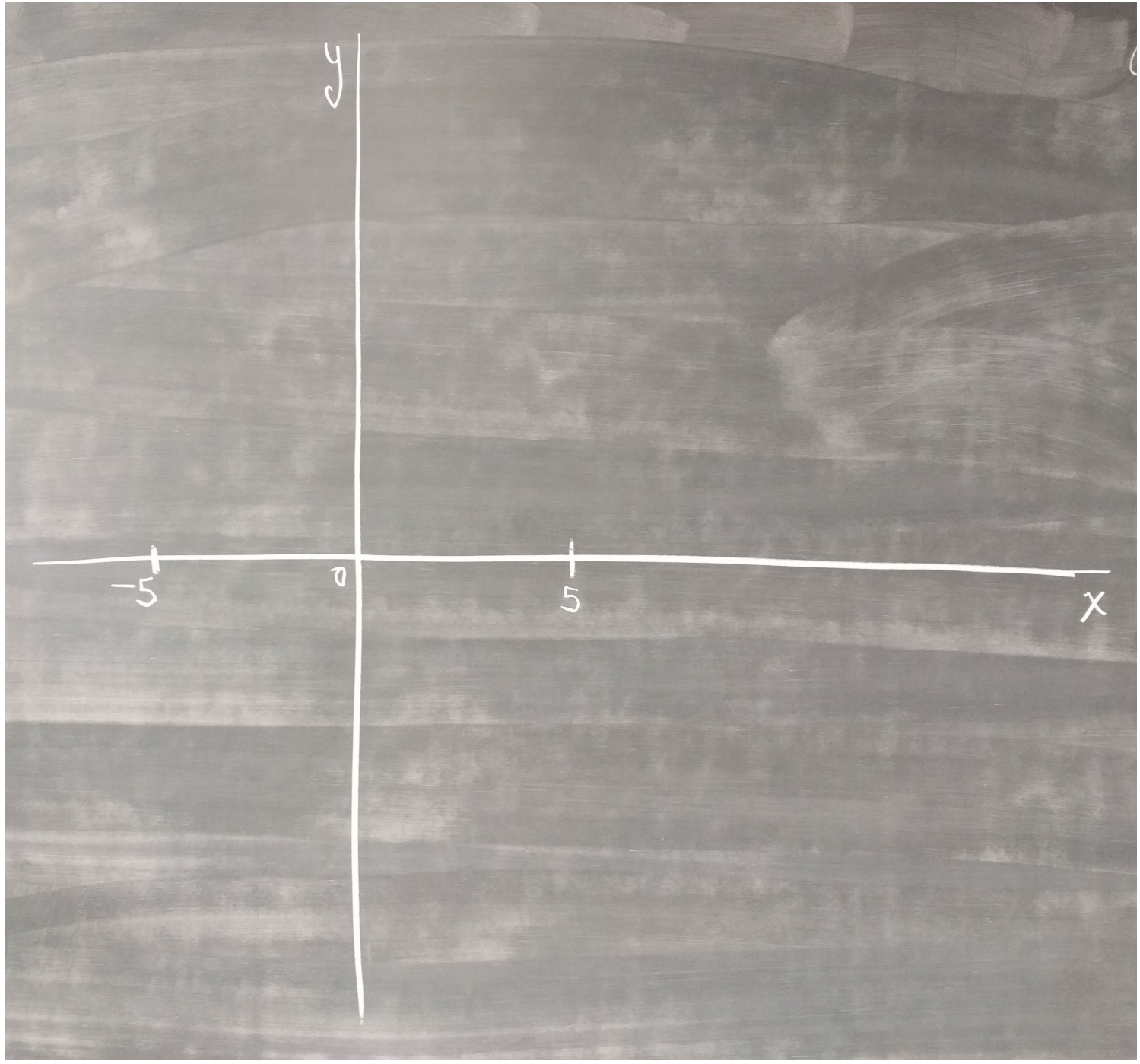
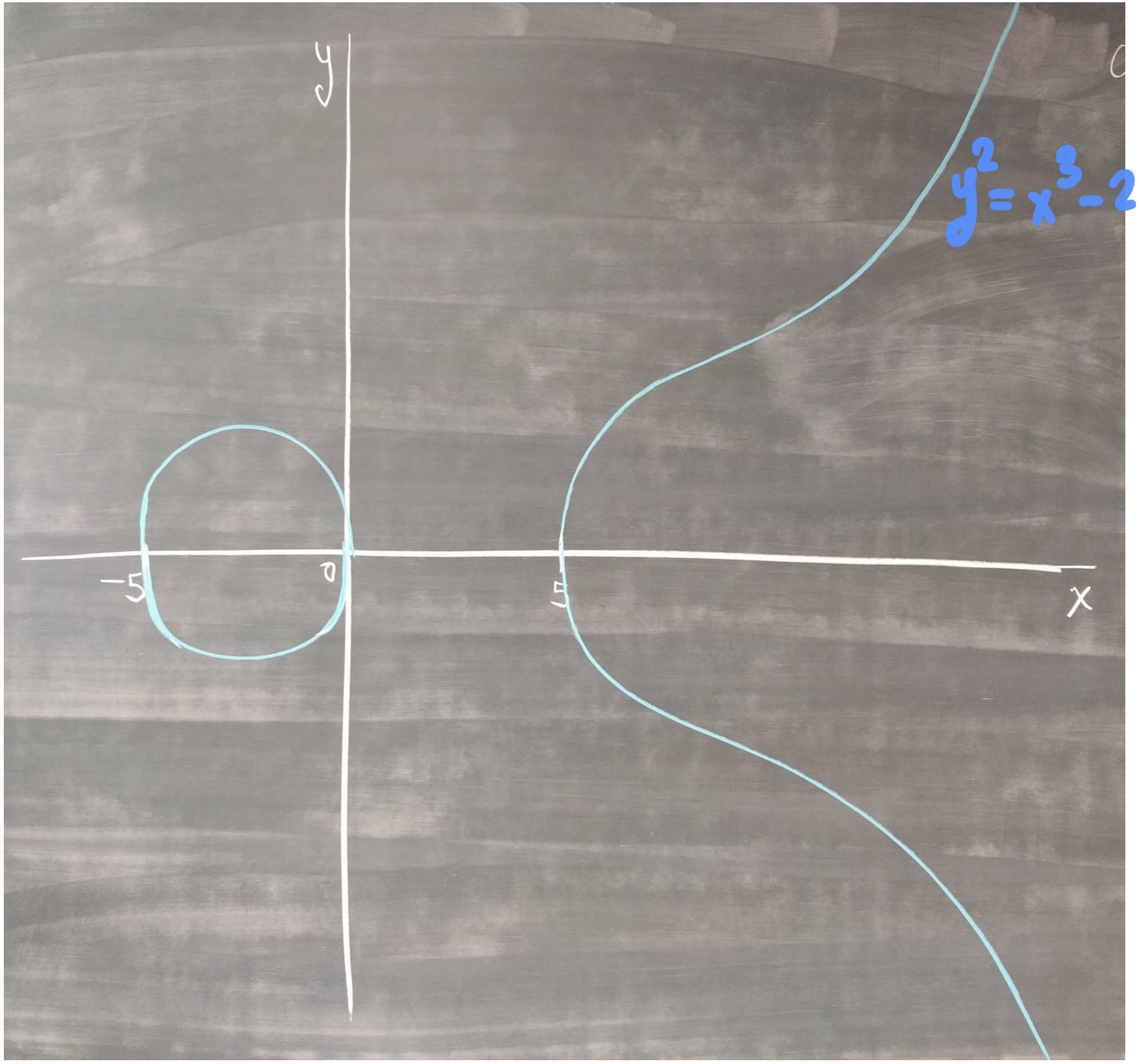
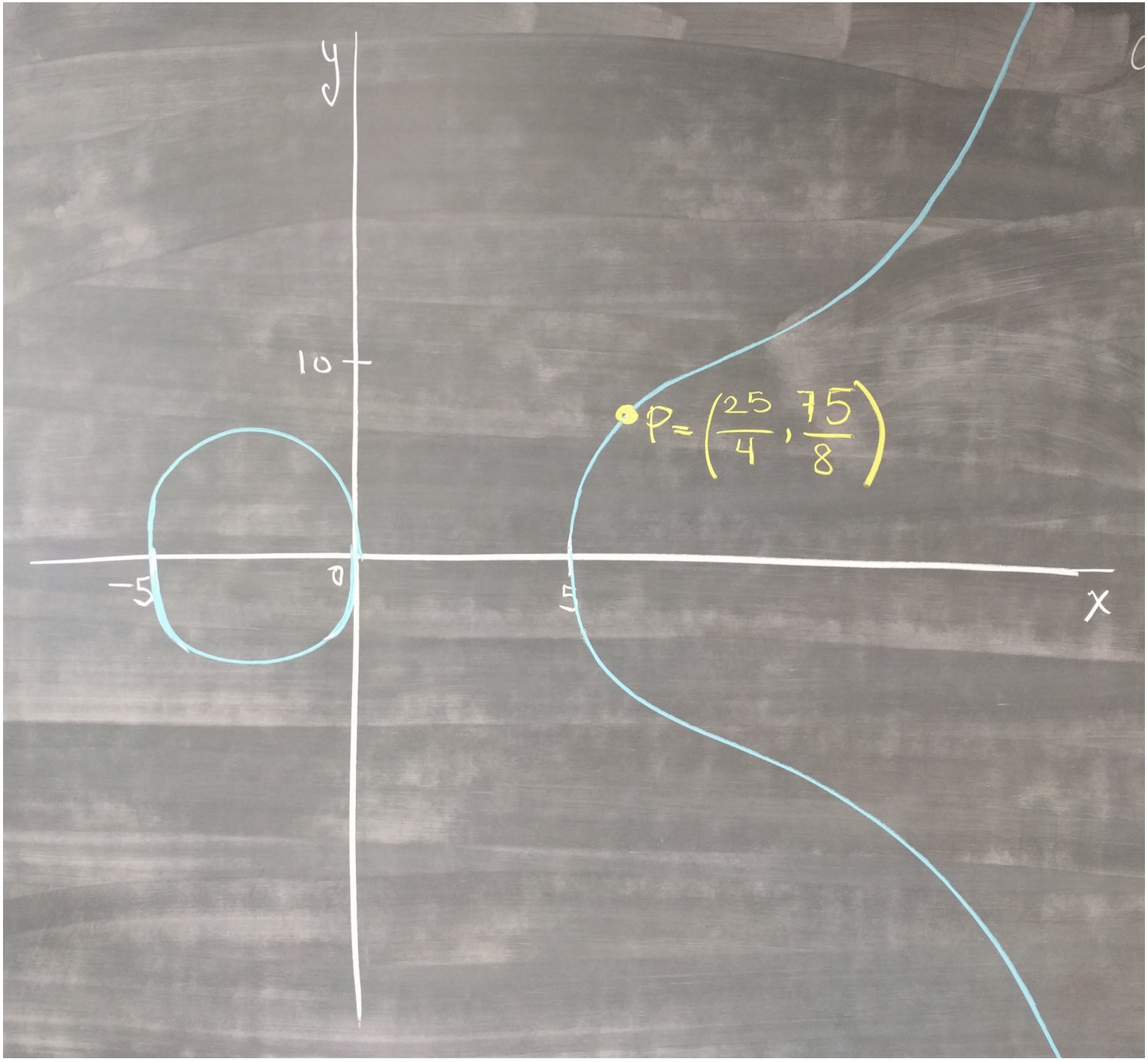
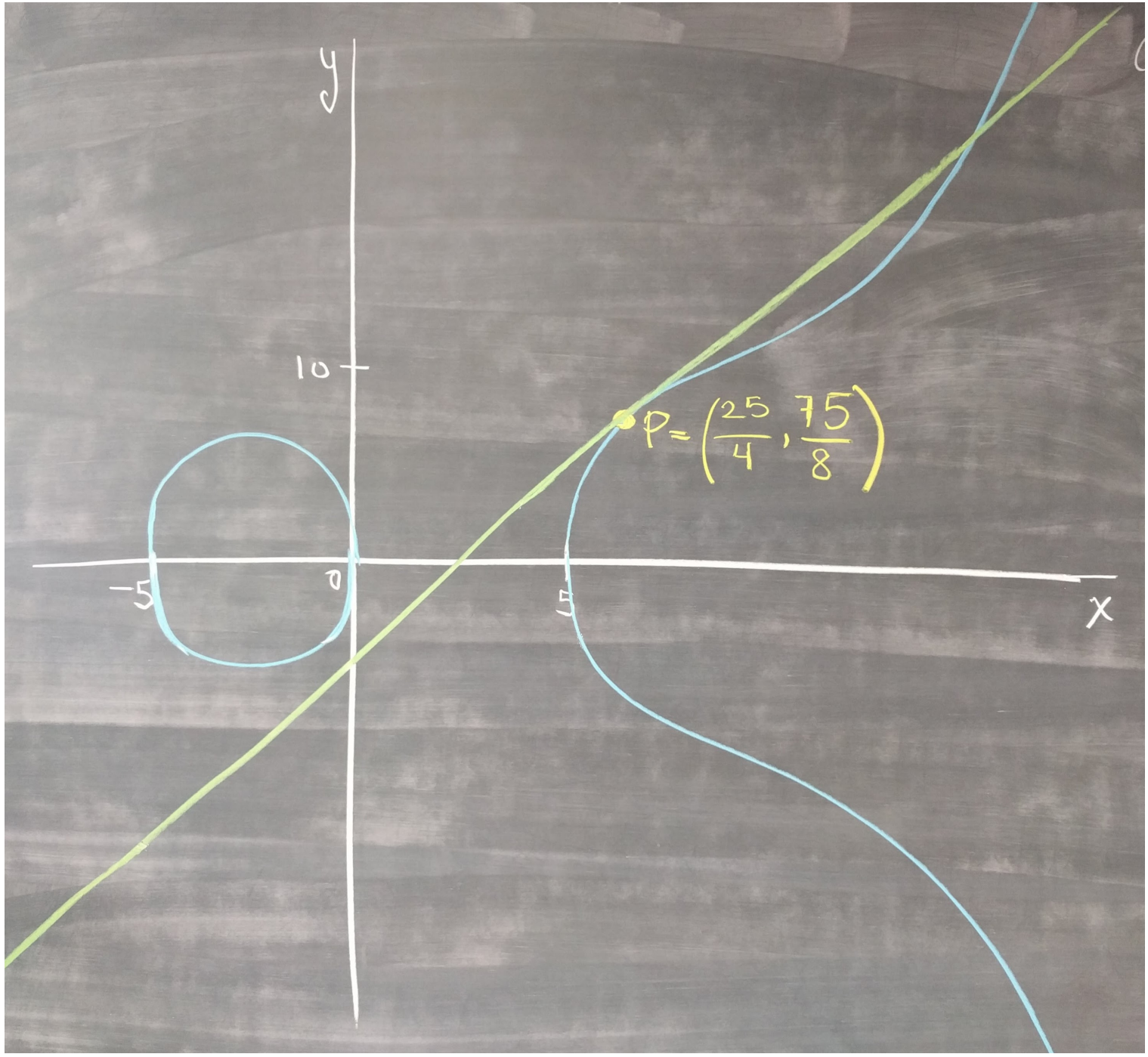
Let's **grab some chalk** and use the theory of elliptic curves to find another right triangle of area 5.


y
x


y
-5
0
5
x


y
x
-5
0
5
$y^2 = x^3 - 25x$


y
10
-5
0
5
x
P=(25/4, 75/8)


y
10
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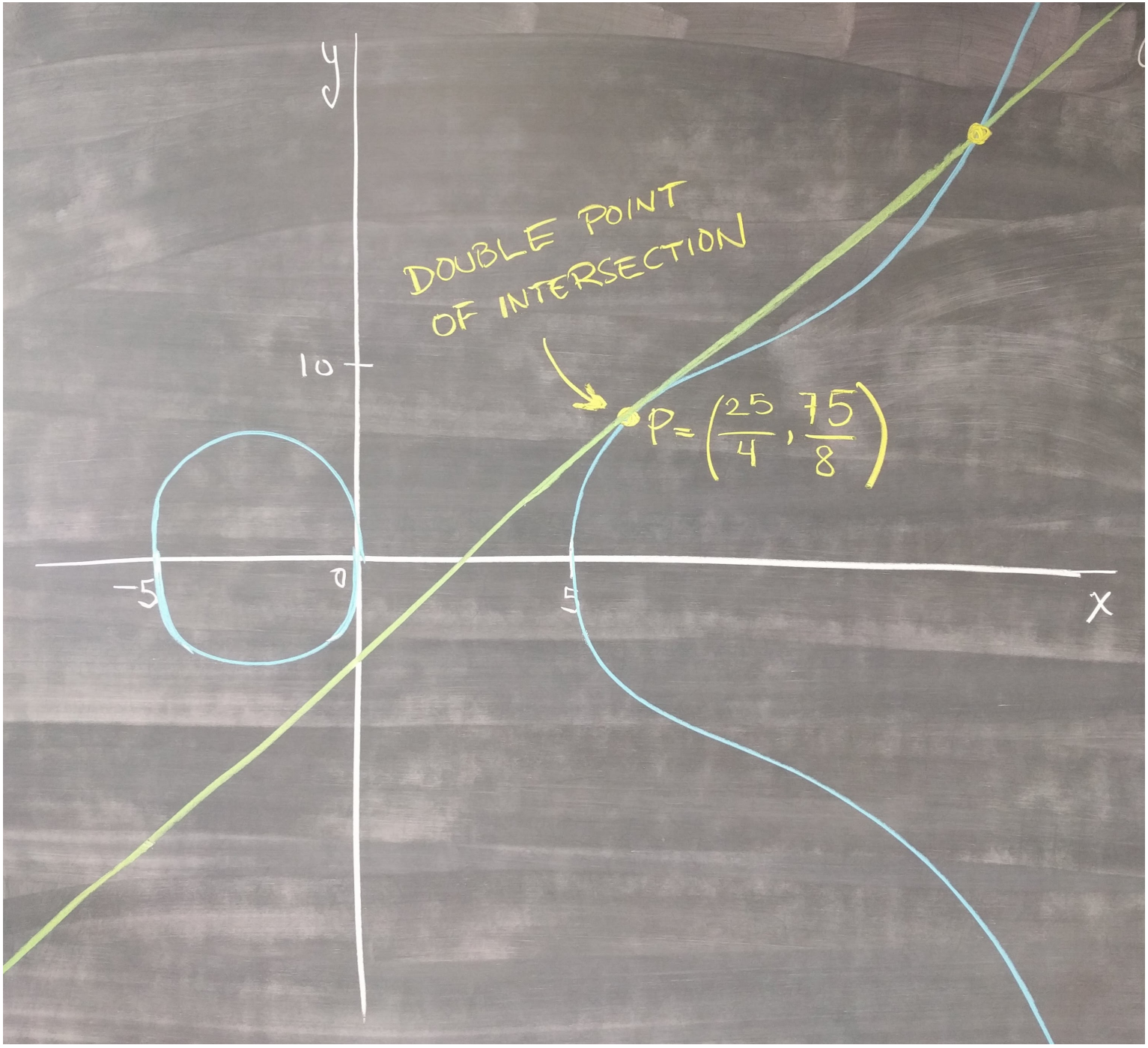
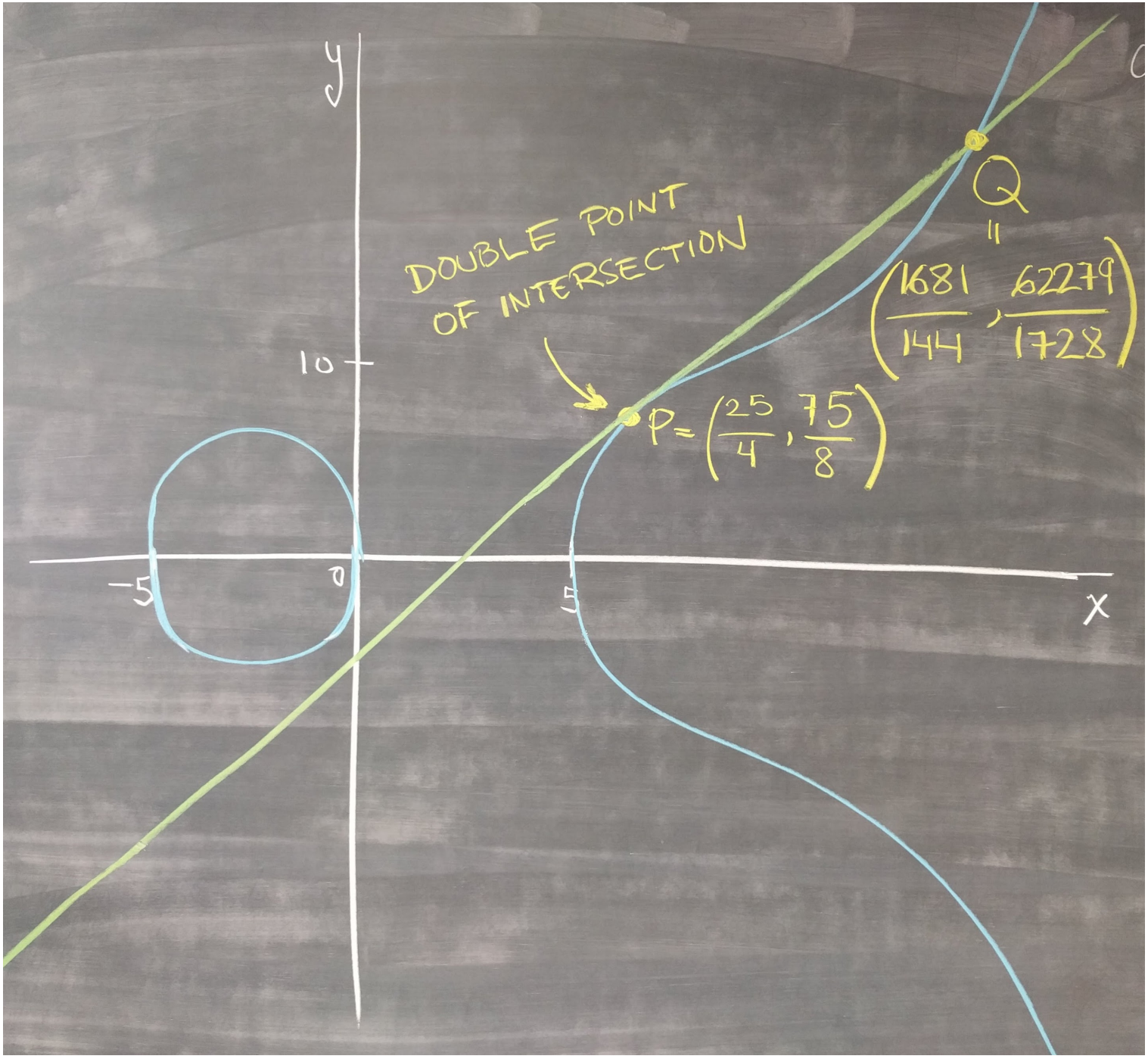
x
P = (25/4, 75/8)


y
10
DOUBLE POINT
OF INTERSECTION
P = (25/4, 75/8)
-5
0
5
x


y
10
-5
0
5
x
DOUBLE POINT
OF INTERSECTION
P = (25/4, 75/8)
Q = (1681/144, 62279/1728)

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Example

The point $P = (\frac{1681}{144}, \frac{62279}{1728})$ on the curve $y^2 = x^3 - 25x$ corresponds to the triangle

$$\left(\frac{1519}{492}, \frac{4920}{1519}, \frac{3344161}{747348}\right)$$

of area 5.

For a fixed $n \geq 1$, the curve $y^2 = x^3 - n^2 x$ is an example of an elliptic curve.

Definition

An elliptic curve E over a field F is a (projective) smooth cubic curve (genus one), with at least one point defined over F.

We are interested in determining all F-rational points on E:

$$E(F) = \{(x_0, y_0) \in E : x_0, y_0 \in F\} \cup \{\mathcal{O} = [0 : 1 : 0]\}.$$

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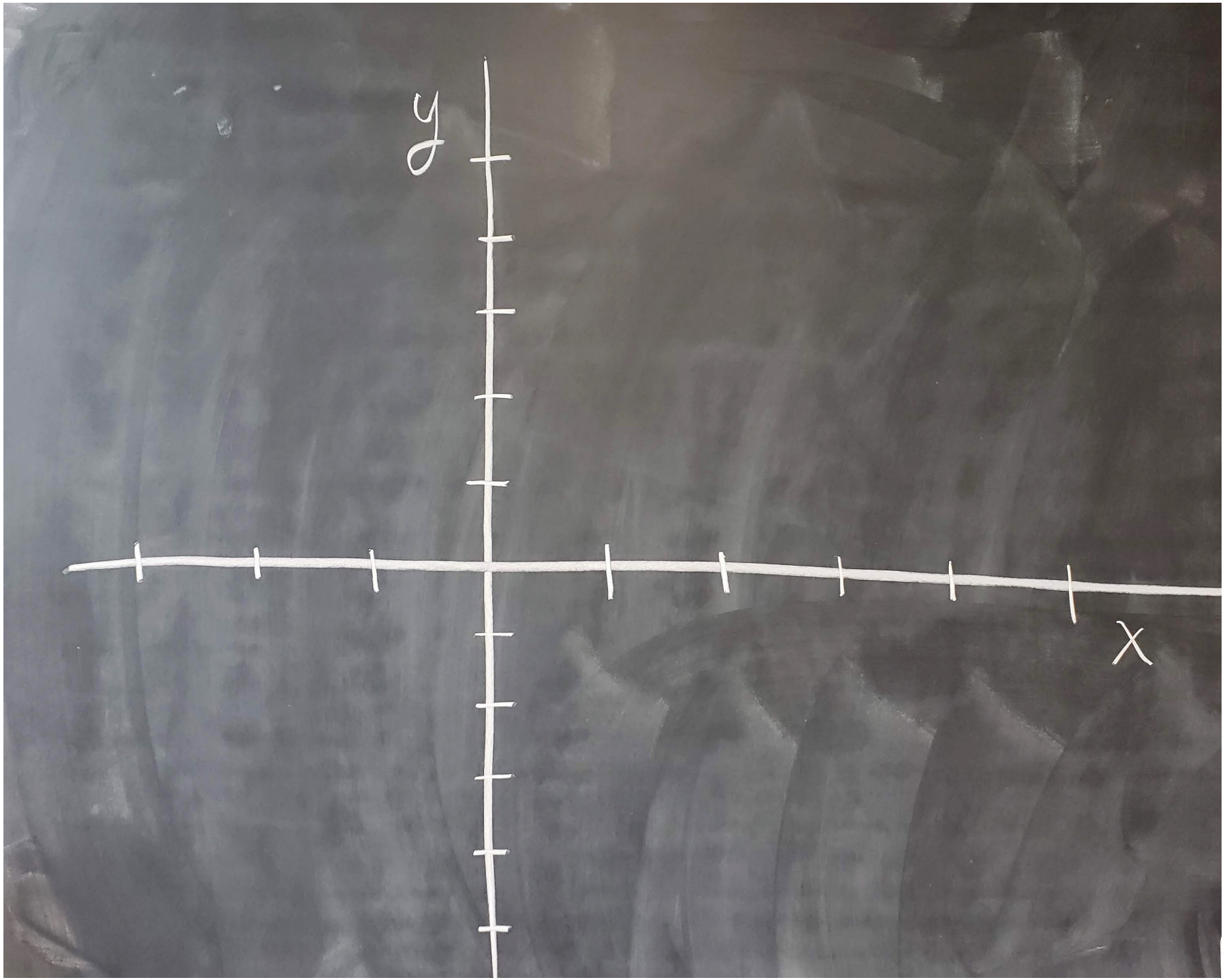
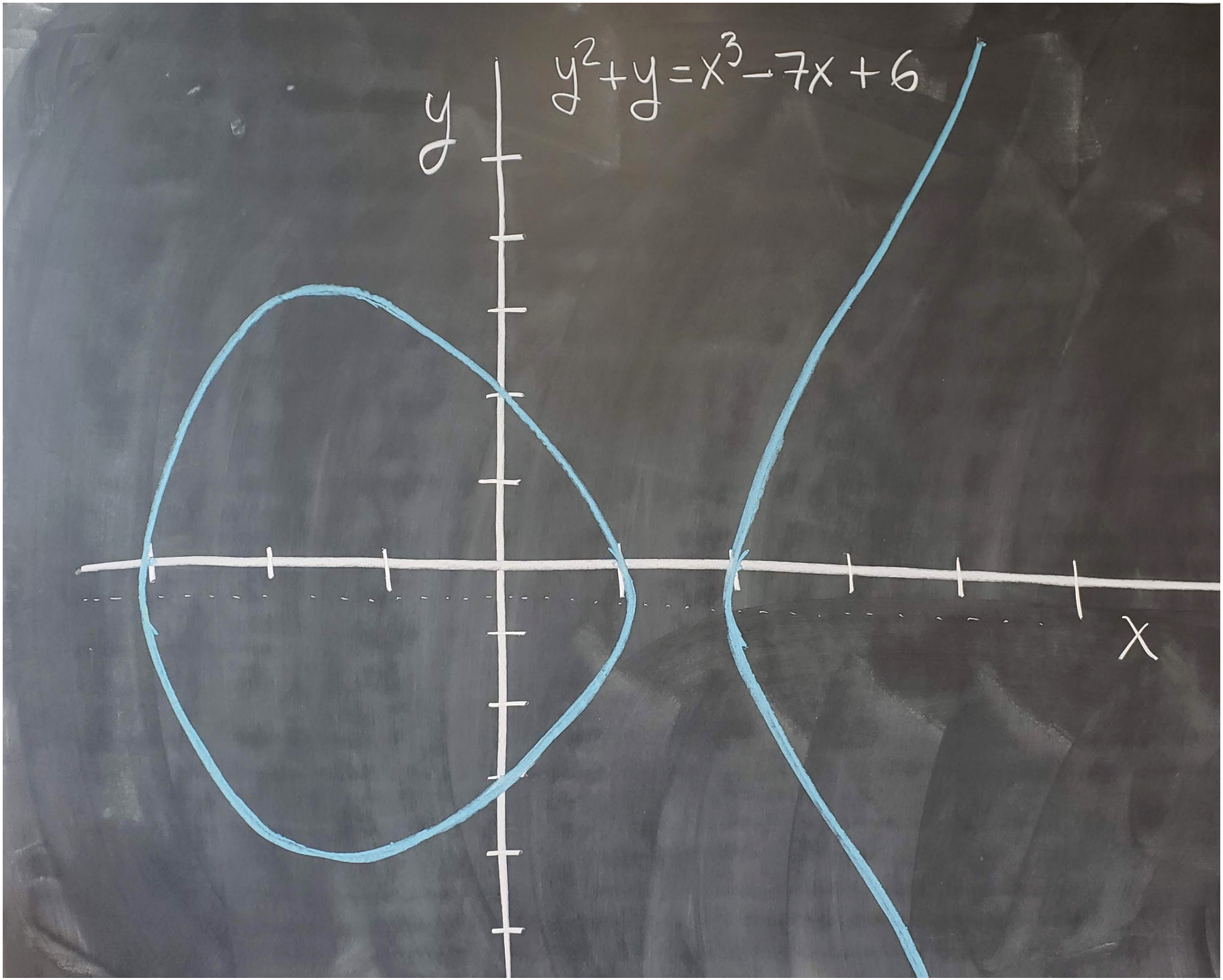
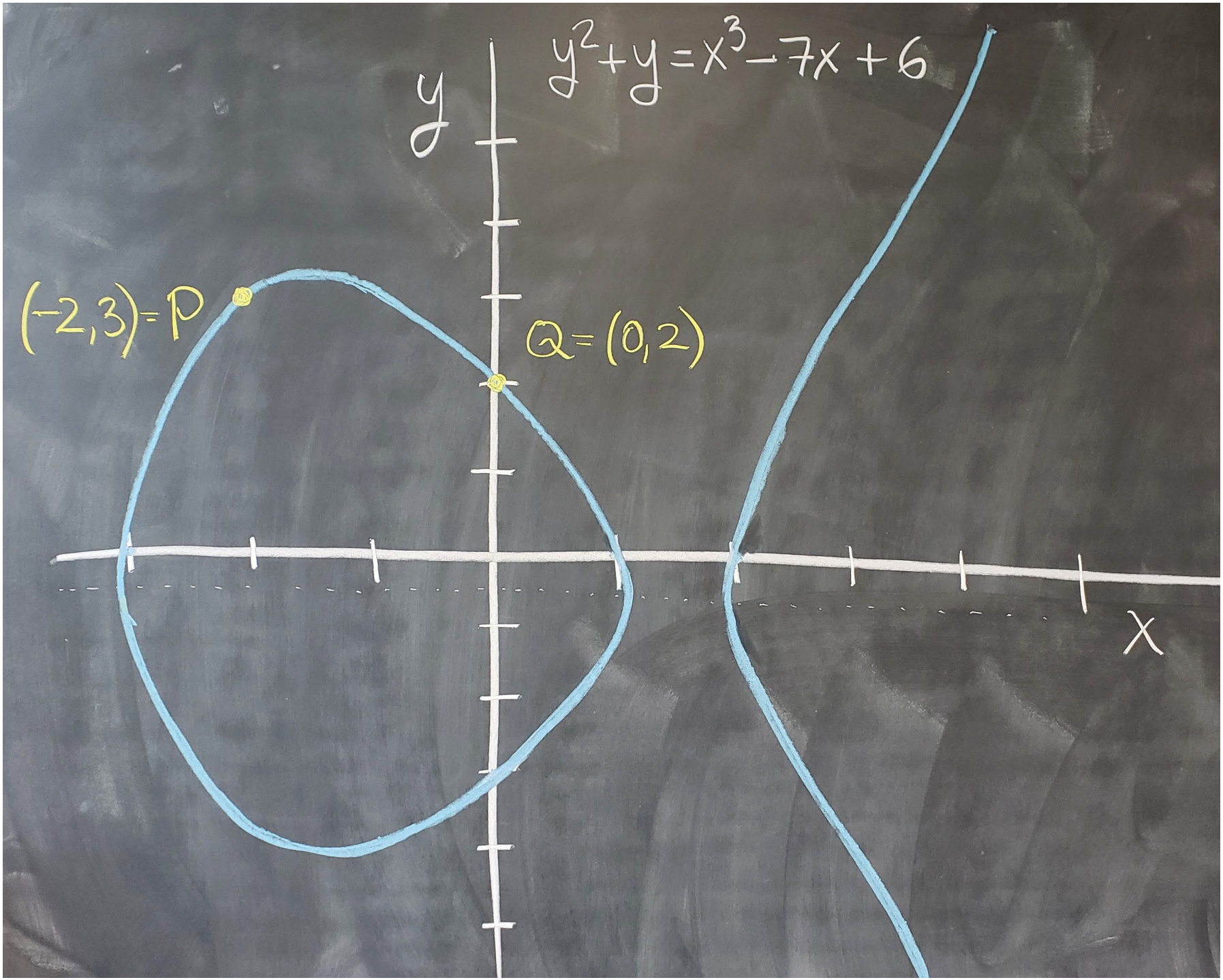
KEY FEATURE OF ELLIPTIC CURVES:

The set of F-rational points $E(F)$ of an elliptic curve E/F can be endowed with a group structure, defined geometrically (also algebraically through groups of divisors).


y
x


$y^2+y=x^3-7x+6$
y
x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$(-2,3)=P$
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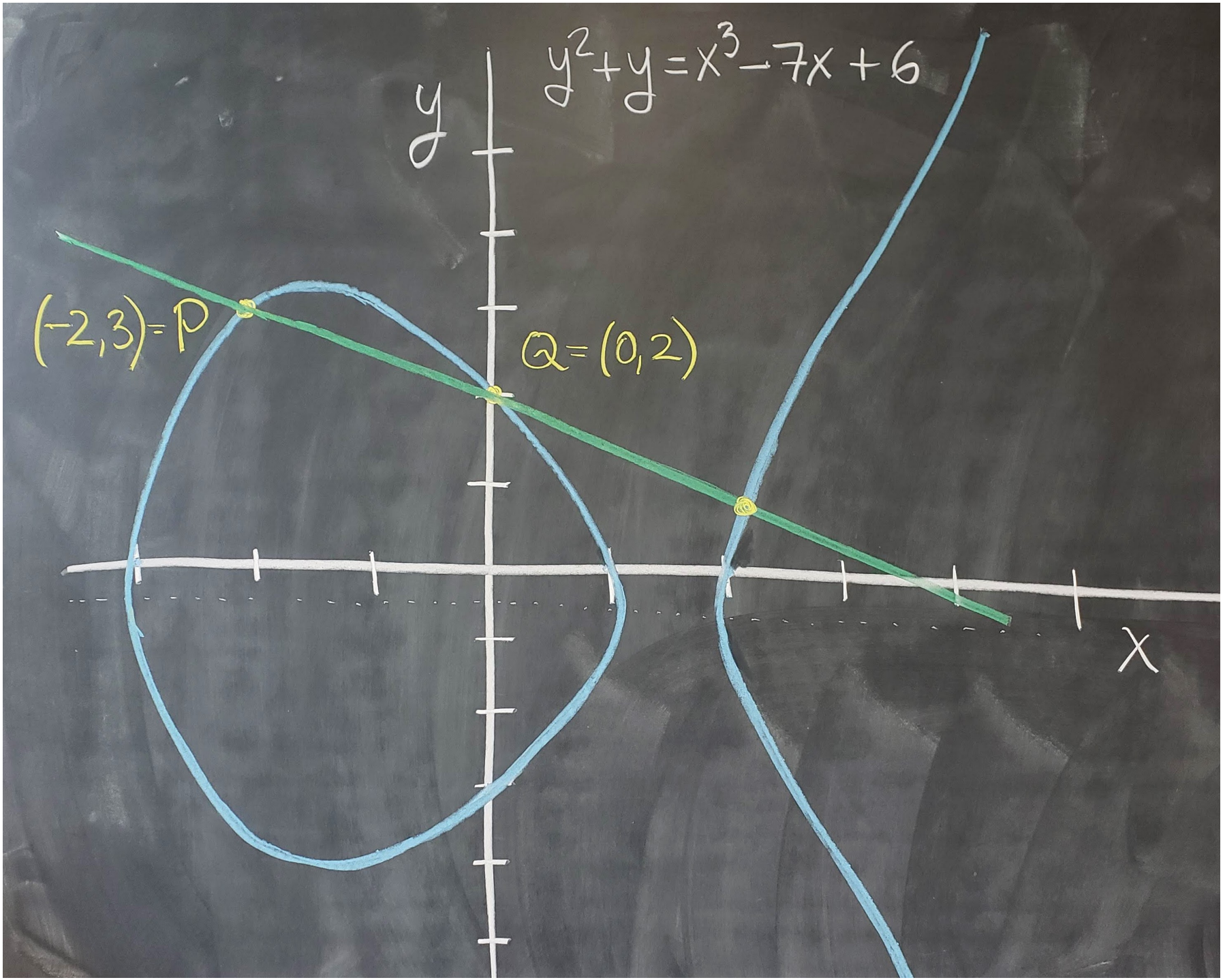
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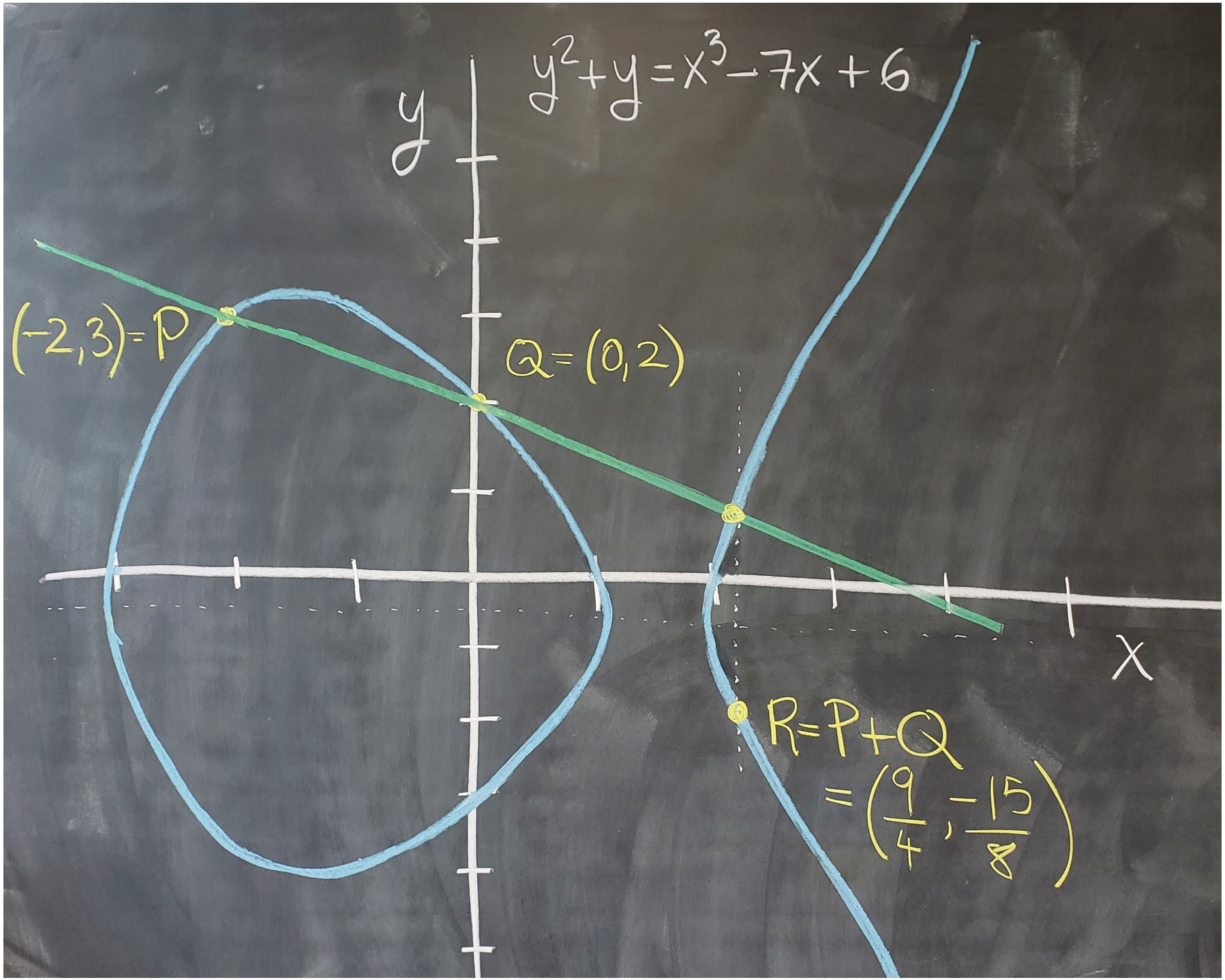
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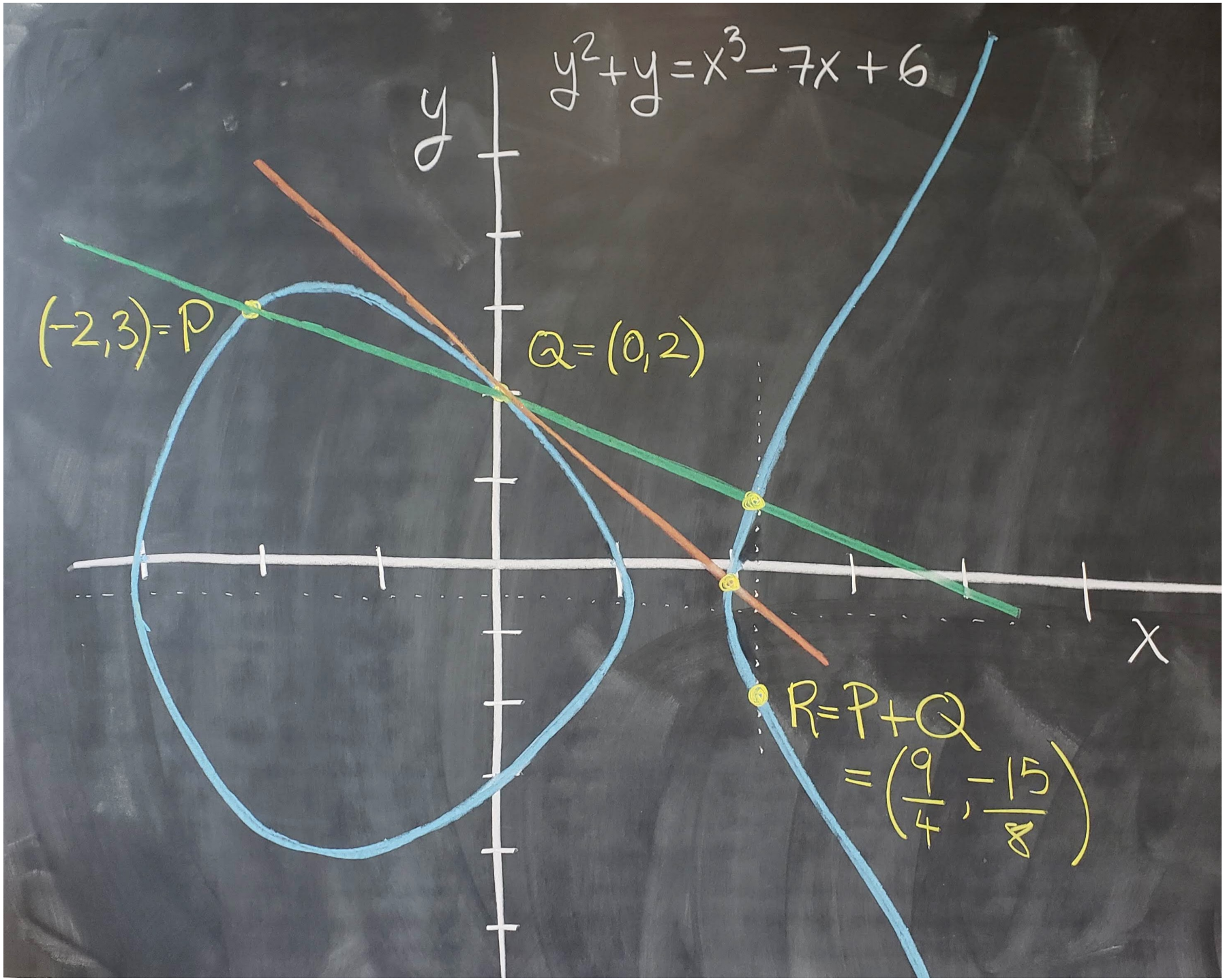
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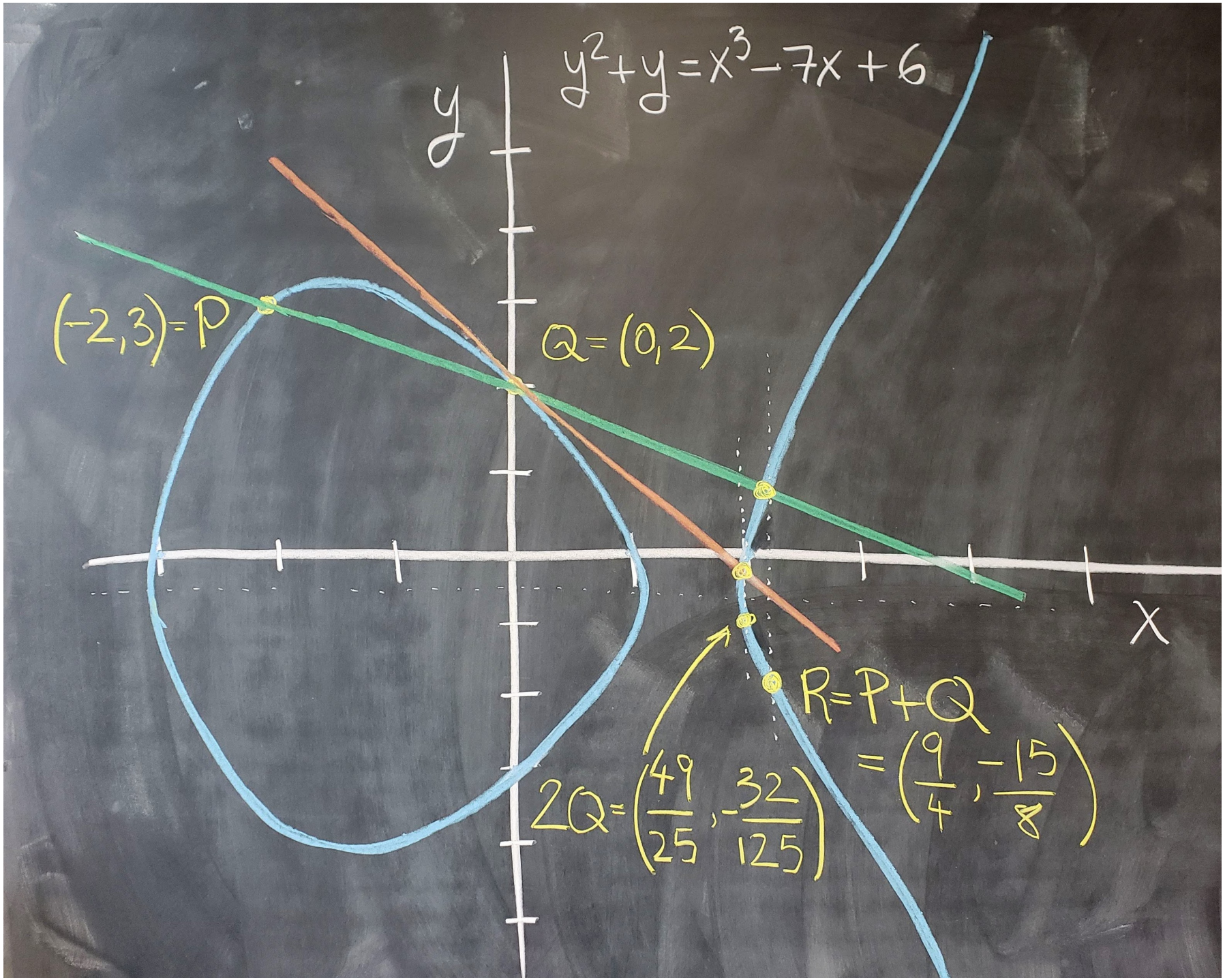
$y^2+y=x^3-7x+6$
y
x
$(-2,3)=P$
$Q=(0,2)$
$2Q=\left(\frac{49}{25},-\frac{32}{125}\right)$
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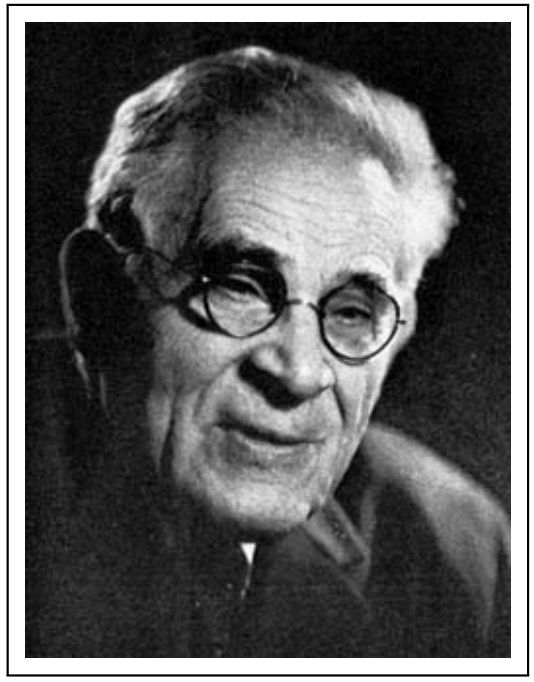
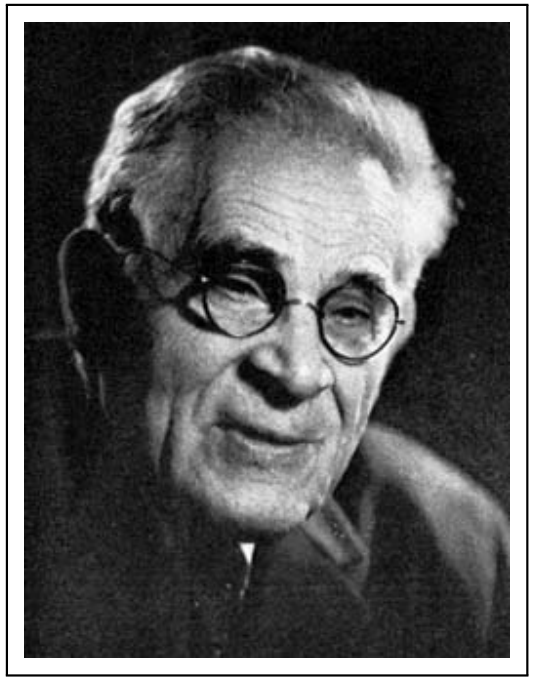
$=\left(\frac{9}{4},\frac{-15}{8}\right)$



Louis Mordell
$1888 - 1972$

Theorem (Mordell, 1922)

Let $E/\mathbb{Q}$ be an elliptic curve. Then, the group of $\mathbb{Q}$-rational points on E, denoted by $E(\mathbb{Q})$, is a finitely generated abelian group. In particular, $E(\mathbb{Q}) \cong E(\mathbb{Q})_{tors} \oplus \mathbb{Z}^{R_{E/\mathbb{Q}}}$ where $E(\mathbb{Q})_{tors}$ is a finite subgroup, and $R_{E/\mathbb{Q}} \geq 0$.



Louis Mordell
1888 – 1972

André Weil
1906 – 1998

Theorem (Mordell–Weil, 1928)

Let F be a number field, and let A/F be an abelian variety. Then, the group of F-rational points on A, denoted by $A(F)$, is a finitely generated abelian group. In particular, $A(F) \cong A(F)_{tors} \oplus \mathbb{Z}^{R_{A/F}}$ where $A(F)_{tors}$ is a finite subgroup, and $R_{A/F} \geq 0$.

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$$E_2(\mathbb{Q}) = \{\mathcal{O}, (2, \pm 3), (0, \pm 1), (-1, 0)\} \cong \mathbb{Z}/6\mathbb{Z}.$$

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3. The curve $E_3/\mathbb{Q} : y^2 = x^3 - 2$ does not have any rational torsion points other than $\mathcal{O}$. However, $E_3(\mathbb{Q}) = \langle (3, 5) \rangle \cong \mathbb{Z}$.

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4. The elliptic curve $E_4/\mathbb{Q} : y^2 = x^3 + 7105x^2 + 1327104x$ features both torsion and infinite order points. In fact, $E_4(\mathbb{Q}) \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}^3$. The torsion subgroup is generated by the point of order 4 $T = (1152, 111744)$. The free part is generated by

$$P_1 = (-6912, 6912), P_2 = (-5832, 188568), P_3 = (-5400, 206280).$$

$$E(\mathbb{Q}) \cong E(\mathbb{Q})_{\text{tors}} \oplus \mathbb{Z}^{R_{E/\mathbb{Q}}}$$

What torsion subgroups $E(\mathbb{Q})_{\text{tors}}$ are possible?

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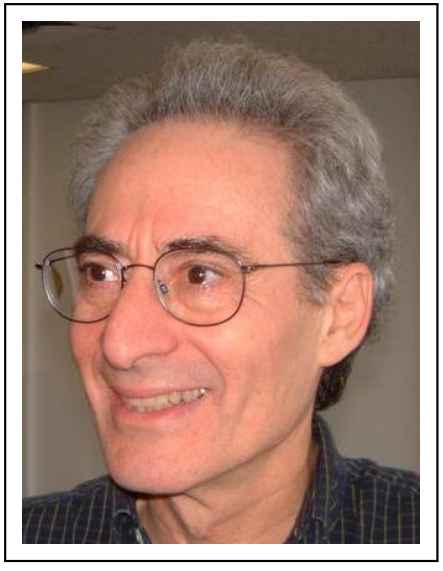


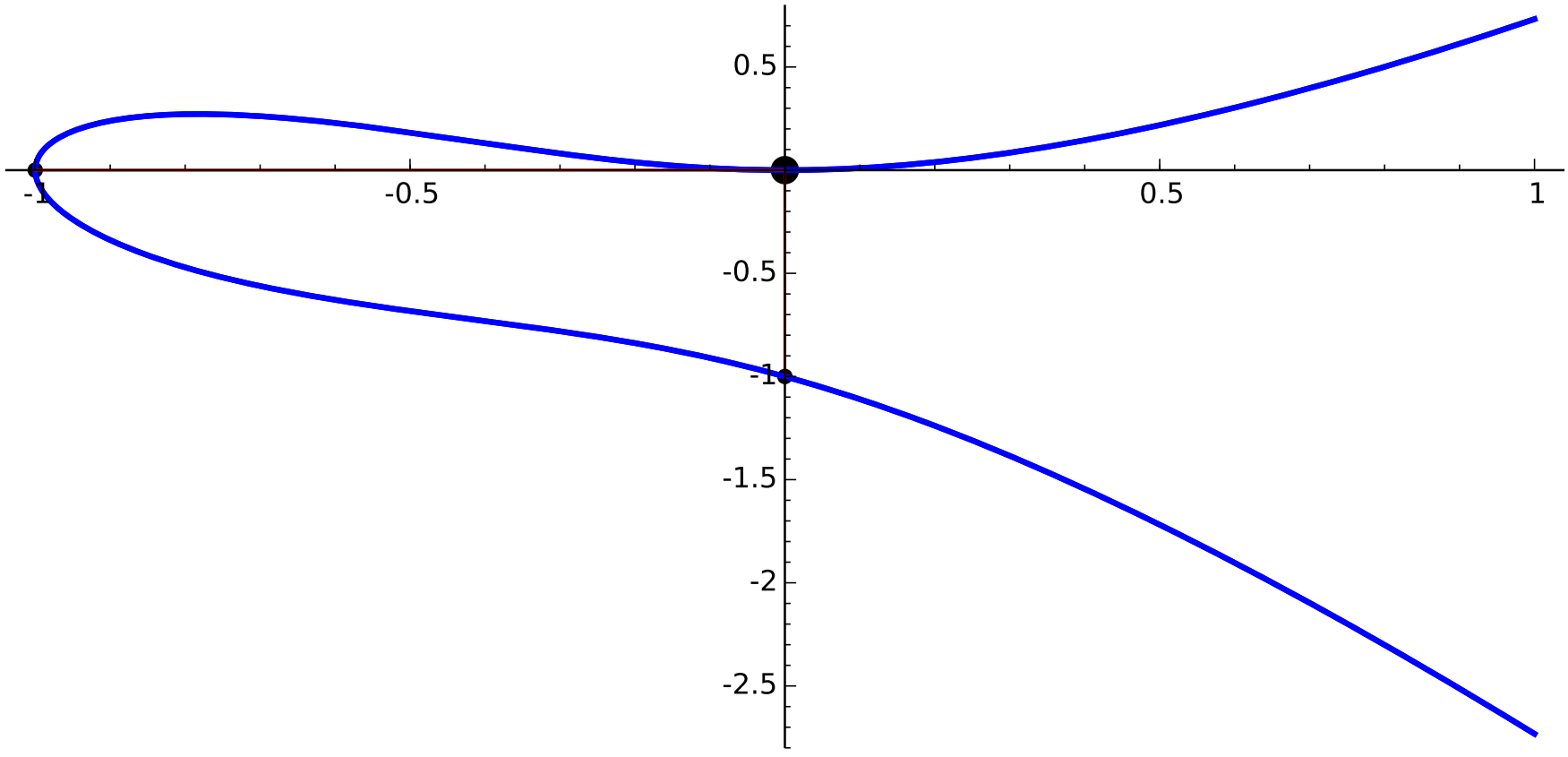
Barry Mazur

Theorem (Levi–Ogg Conjecture; Mazur, 1977)

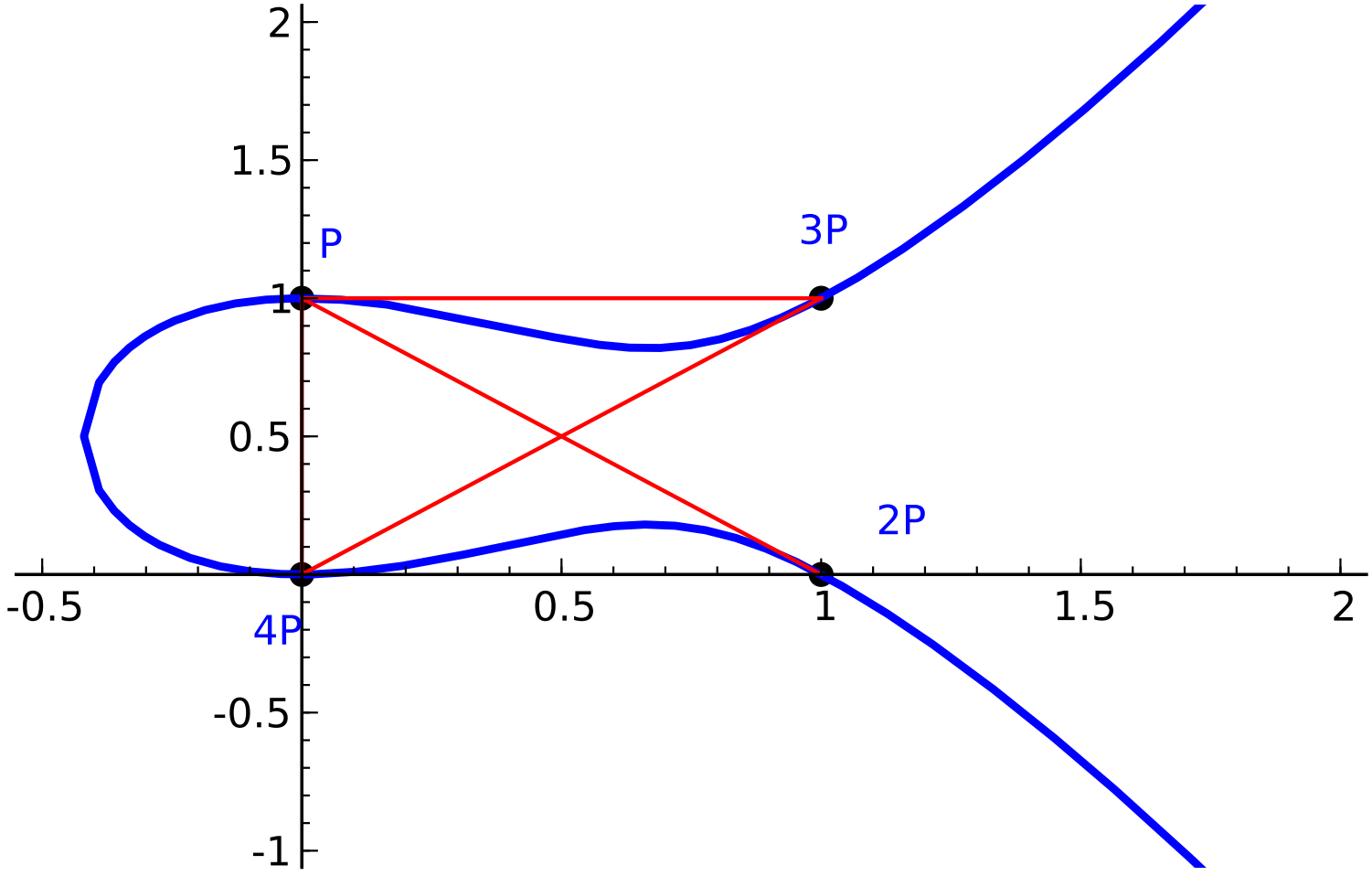
Let $E/\mathbb{Q}$ be an elliptic curve. Then

$$E(\mathbb{Q})_{tors} \simeq \begin{cases} \mathbb{Z}/M\mathbb{Z} & \text{with } 1 \leq M \leq 10 \text{ or } M = 12, \text{ or} \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2M\mathbb{Z} & \text{with } 1 \leq M \leq 4. \end{cases}$$

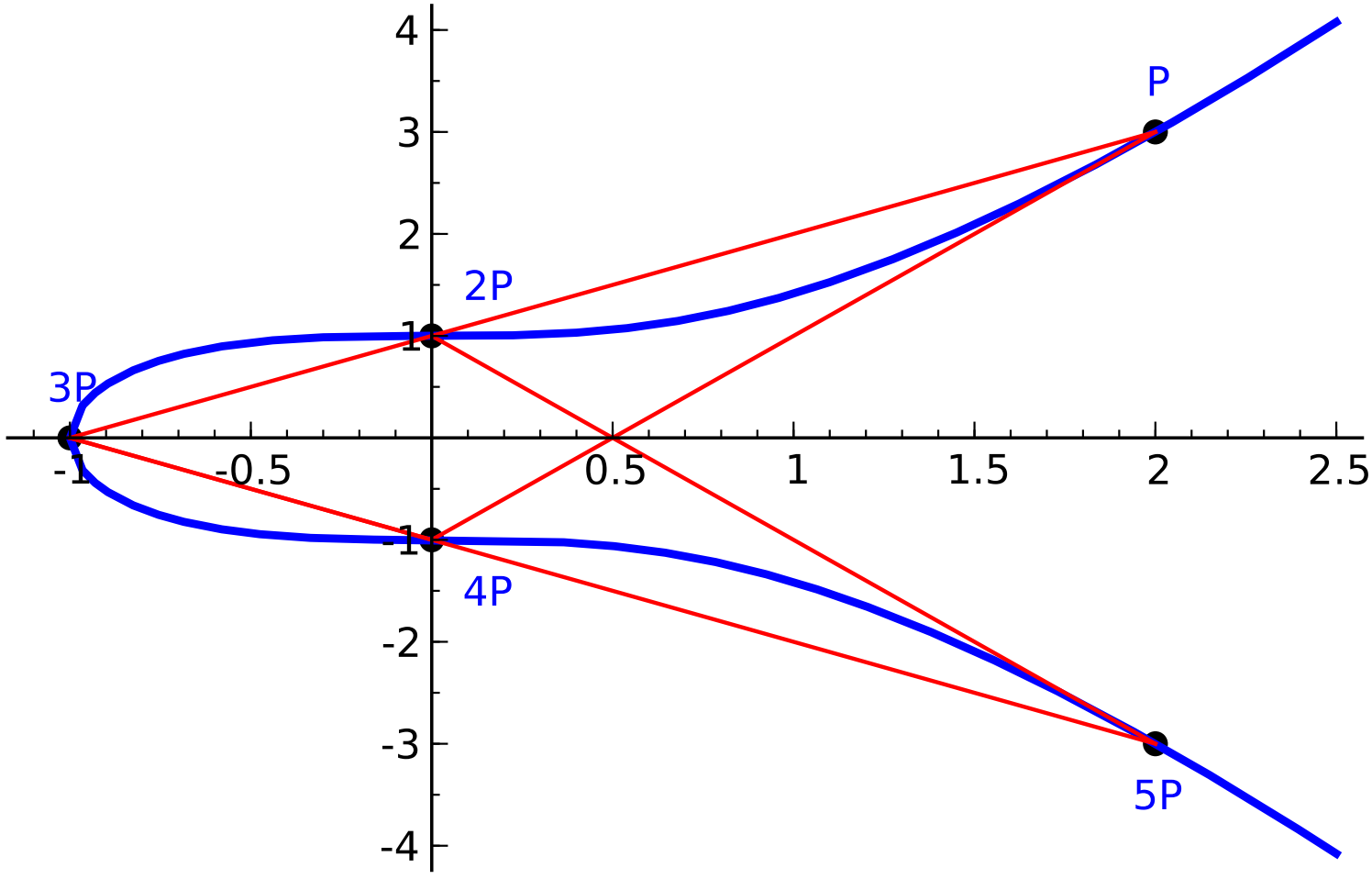
Moreover, each possible group appears infinitely many times.



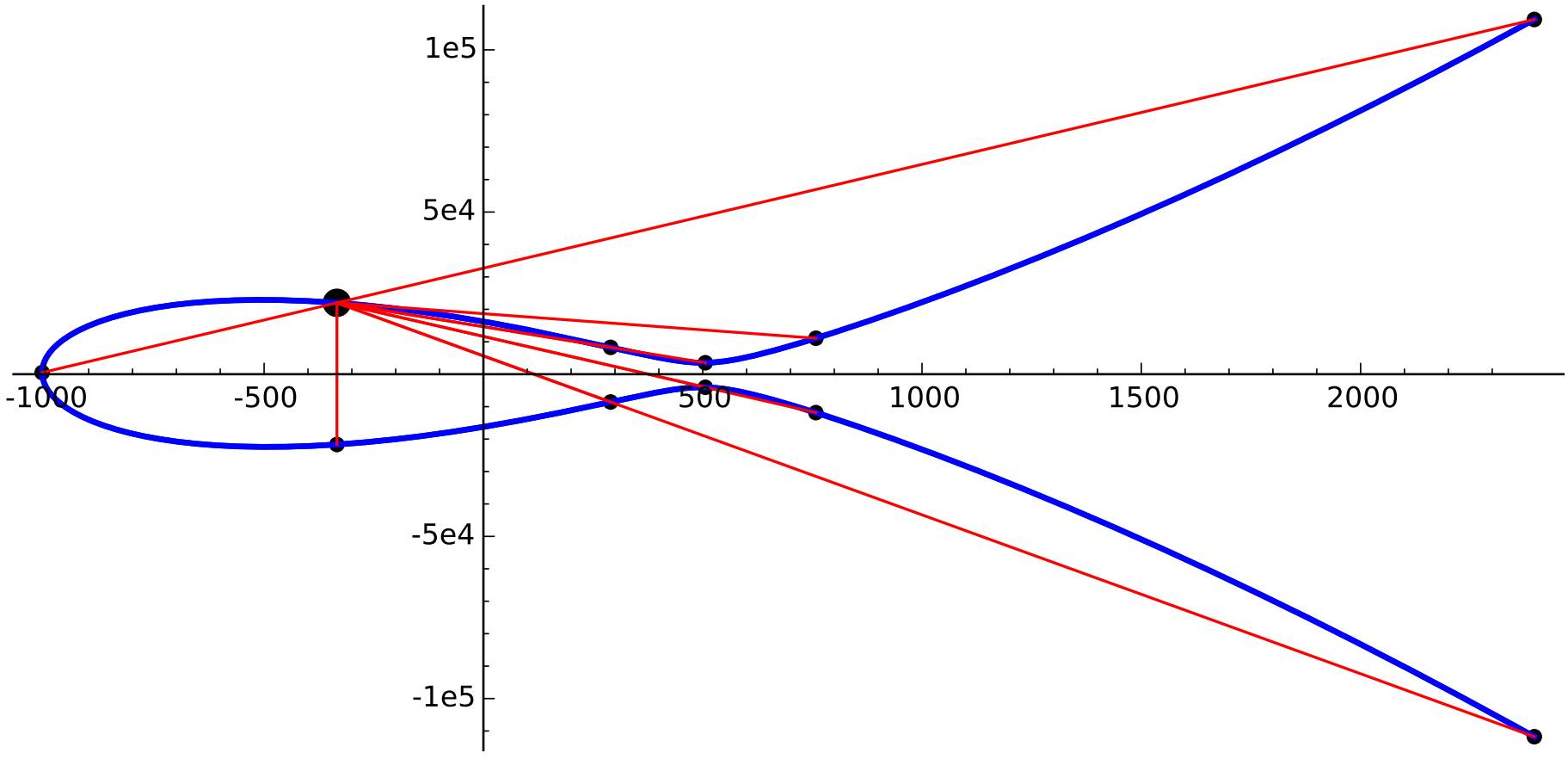
The elliptic curve $E/\mathbb{Q} : y^2 + xy + y = x^3 + x^2$
has a point $P = (0, 0)$ of order 4.



The curve $E/\mathbb{Q} : y^2 - y = x^3 - x^2$ has a point $P = (0, 1)$ of order 5.



The elliptic curve $E/\mathbb{Q} : y^2 = x^3 + 1$ has a point $P = (2, 3)$ of order 6.



The elliptic curve `30030bt1` has a point of order 12.

$$y^2 + xy = x^3 - 749461x + 263897441$$

"Torsion Groups and Galois Representations of Elliptic Curves"
Zagreb (Croatia), June 25-29, 2018.

$$E(\mathbb{Q}) \cong E(\mathbb{Q})_{\text{tors}} \oplus \mathbb{Z}^{R_{E/\mathbb{Q}}}$$

What ranks $R_{E/\mathbb{Q}}$ of elliptic curves over $\mathbb{Q}$ are possible?

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What ranks $R_{E/\mathbb{Q}}$ of elliptic curves over $\mathbb{Q}$ are possible?

Open Problem

What values can $R_{E/\mathbb{Q}}$ take? In particular, can $R_{E/\mathbb{Q}}$ be arbitrarily large, or is it uniformly bounded?

Example (2006): Elkies' elliptic curve of rank ≥ 28 ($= 28$ under GRH!)

$$y^2 + xy + y = x^3 - x^2 - (20067762415575526585033208209338542750930230312178956502)x + (3448161179503055646703298569039072037485594435931918036126600829629193944873224342 9)$$

Noam Elkies

Independent points of infinite order:

$$P_1 = [-212415009125438107329213746 3, 259854492051899599030515511070780628911531]$$

$$P_2 = [233450986603470175688475453 7, 188720041954944691808683165528036279315 31]$$

$$P_3 = [-167173605406236906387903866 3, 251709377261144287808506947241319126049131]$$

$$\vdots$$

$$
\begin{aligned}
P_4 =&[213913026013915666492982137,\\
&366395091714397292024214596929412975275 31]\\
P_5 =&[153470676446712072388547733 7,\\
&854295853460176942890210328627810727995 31]\\
P_6 =&[-27310794878756770333415750 63,\\
&26252181548433219164128407262390214338753 1]\\
P_7 =&[27757262668445716497054585 37,\\
&12845755474014060248869487699082640369931]\\
P_8 =&[14943857293271889575418338 17,\\
&88486605527733405986116494514049233411451]\\
P_9 =&[18684382286208873585090652 57,\\
&59237403214437708712725140393059358589131]\\
P_{10} =&[20089451088257437748665425 37,\\
&47690677880125552882151750781541424711531]\\
P_{11} =&[23483605409180251696516329 37,\\
&17492930006200557857340332476448043635 31]
\end{aligned}
$$

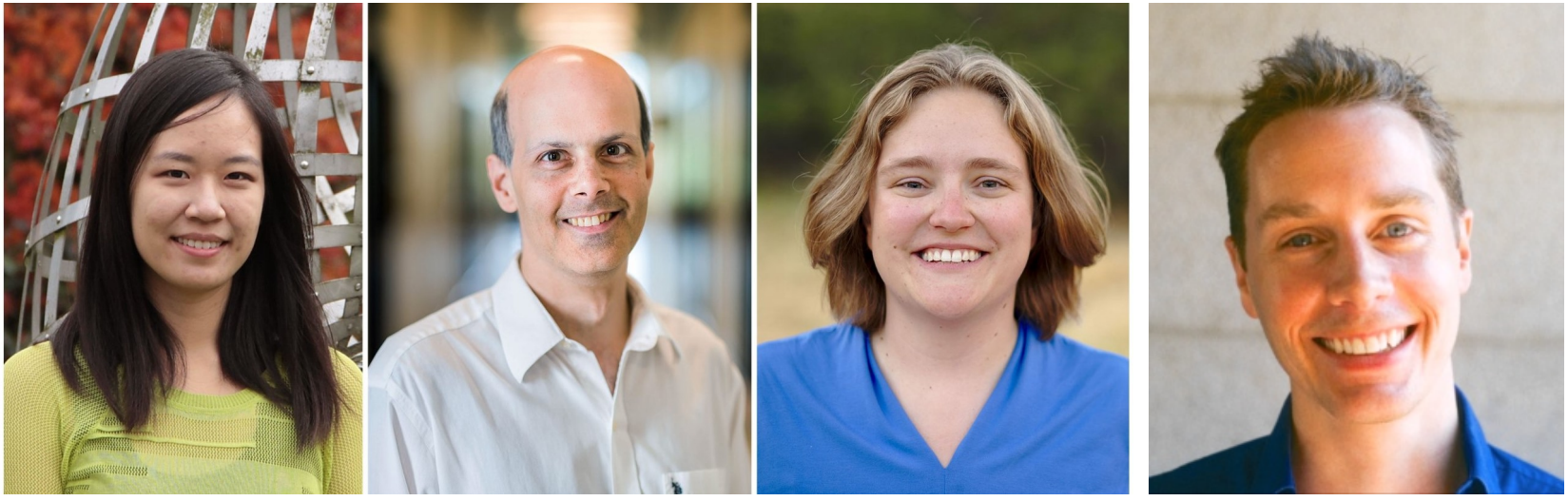
$P12$ = [-147208400709048117447000866 3, 24664345065350371419994744154975979846913 1]
P13 = [29241286077080612133632889 37, 28350264431488878501488356474767375899531]
P14 = [53749938910660618932939345 37, 28618890842726338645117503191647989373153 1]
P15 = [17096907682333545233340085 57, 71898834974686089466159700529215980921631]
P16 = [24509540113535931440725951 87, 44452281735326343570492625506107147365 31]
P17 = [29692547092735591674646749 37, 32766893075366270801333682543160469687531]
P18 = [27119149349416926013328829 37, 20684366127783816986504139815065906135 31]
P19 = [20078586077996854528778328937, 27796085411378066046560517256246240300915 31]
P20 = [21580824502407347743178106 97, 34994373401964026809969662241800901254731]
P21 = [20046454582470590224032249 37, 48049329780704645522439866999888475467531]
P22 = [29757494509479962649470913 37, 33398989826075322320208934410104857869131]
P23 = [-21024904676862851501473478 63, 25957639145987578957167739317168720322753 1]
P24 = [31158317991506303490219453 7, 16810438522998060354010947291566015347393 1]
P25 = [27739310083418652314437718 17, 12632162834649921002414116273769275813451]
P26 = [21565811881437684093634613 87, 35125092964022908897004150516375178087331]
P27 = [38663304998724125088156591 37, 12119775565594422629303692671502584732253 1]
P28 = [22308682897735760237786787 37, 28558760030597485663387020600768640028531]

Open Problem

Can the rank $R_{E/\mathbb{Q}}$ of an elliptic curve be arbitrarily large?

Conjectures and heuristic arguments for and against:

- Néron (1950), Honda (1960): Yes (bounded).
- Cassels (1966), Tate (1974), Mestre (1982), Silverman (1986, 2009), Brumer (1992), Ulmer (2002), Farmer–Gonek–Hughes (2007): No (unbounded).
- Rubin–Silverberg (2000), Granville (2006), Watkins (2015), Park–Poonen–Voight–Wood (2016): Yes (bounded).



Jennifer Park, Bjorn Poonen, Melanie Matchett Wood, John Voight.

Conjecture (Park, Poonen, Voight, Wood)

The ranks $R_{E/\mathbb{Q}}$ are bounded, and there are only finitely many rank values above 21*.*

Goal

Our goal is to understand the possible structures of $E(\mathbb{Q})$, for an elliptic curve $E/\mathbb{Q}$.

$$E(\mathbb{Q}) \cong E(\mathbb{Q})_{\text{tors}} \oplus \mathbb{Z}^{R_{E/\mathbb{Q}}}$$

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The torsion subgroups over $\mathbb{Q}$ are the "poster child" of what an arithmetic group should be like. Torsion subgroups are:

- Computable
- Classified
- Parametrized in families
- Statistically understood

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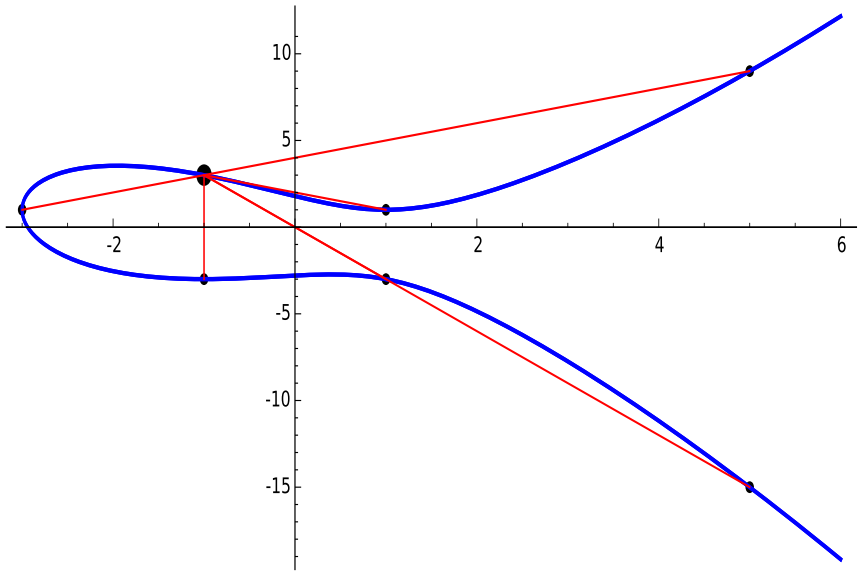

The curve $E : y^2 + xy + y = x^3 + x^2 - 4x + 5$ (`42.a5`) has torsion subgroup $\langle(-1, 3)\rangle \cong \mathbb{Z}/8\mathbb{Z}$.

- **Computable**
  - Nagell–Lutz theorem.
  - Division polynomials.

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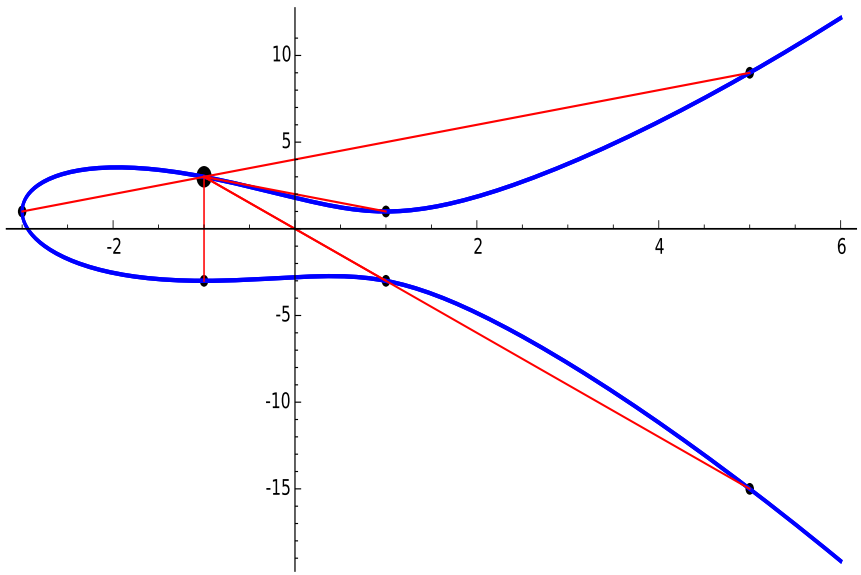

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- **Classified**
  - Mazur's theorem:

$$E(\mathbb{Q})_{\text{tors}} \simeq \begin{cases} \mathbb{Z}/M\mathbb{Z}, \text{ or} \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2N\mathbb{Z} \end{cases}$$

where $1 \leq M \leq 10$ or $M = 12$, and $1 \leq N \leq 4$.

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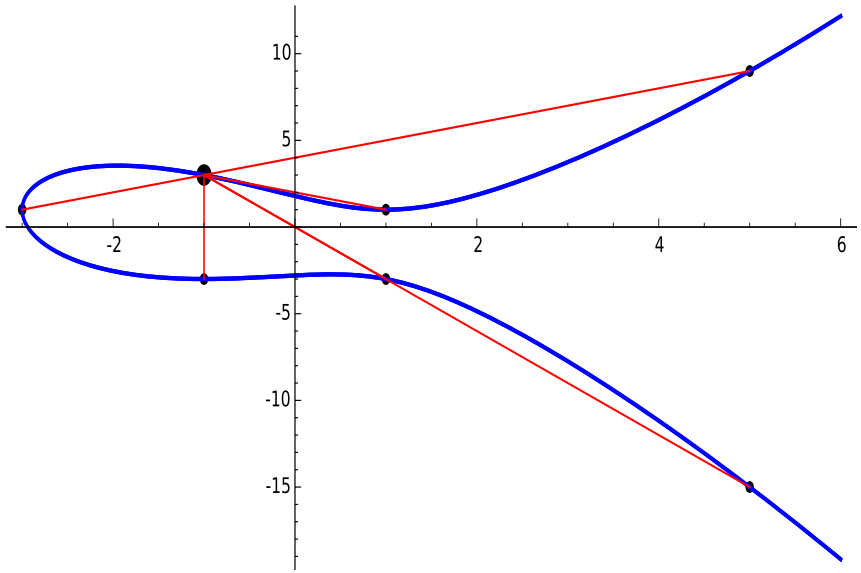

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- **Parametrized in families**
  - Kubert et al.:

e.g.,
elliptic curves with $\mathbb{Z}/8\mathbb{Z}$ tors.:

$$E : y^2+(1-a)xy-by^2 = x^3-bx^2$$

with $b = (2t - 1)(t - 1)$ and $a = b/t$, for any $t \neq 0, 1/2, 1$.

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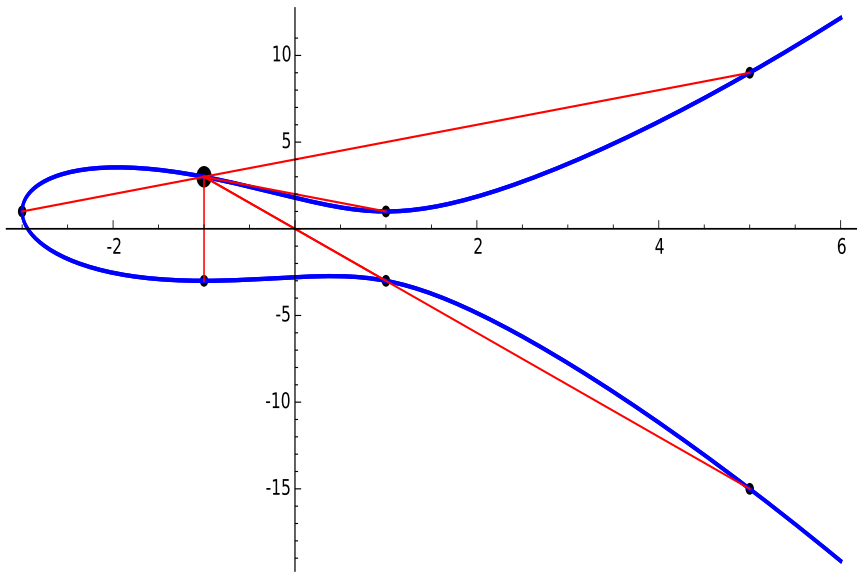

The curve $E' : y^2 + \frac{1}{3}xy - \frac{2}{9}y = x^3 - \frac{2}{9}x^2$ ($\cong_{\mathbb{Q}}$ `42.a5`) has torsion subgroup $\langle(-1,3)\rangle \cong \mathbb{Z}/8\mathbb{Z}$.

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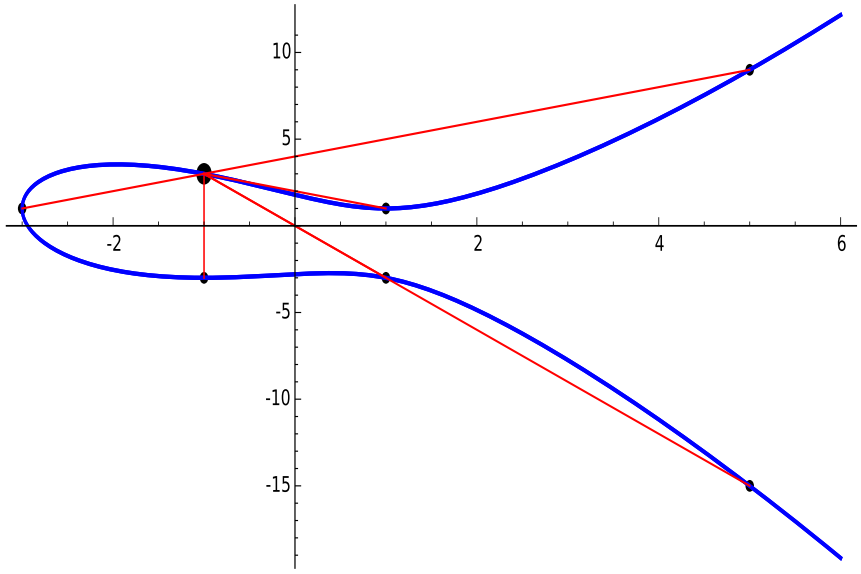

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- **Statistically understood**
  - Harron–Snowden (2013):

  Let $N_G(X)$ be the number of elliptic curves $E/\mathbb{Q}$ with (naive) height $\leq X$ and $E(\mathbb{Q})_{\text{tors}} \cong G$. Then, there are positive constants $C_1, C_2, d(G)$ such that

  $$C_1 X^{d(G)} \leq N_G(X) \leq C_2 X^{d(G)}.$$

  E.g., $d(\{0\}) = 5/6$ and $d(\mathbb{Z}/8\mathbb{Z}) = 1/12$.

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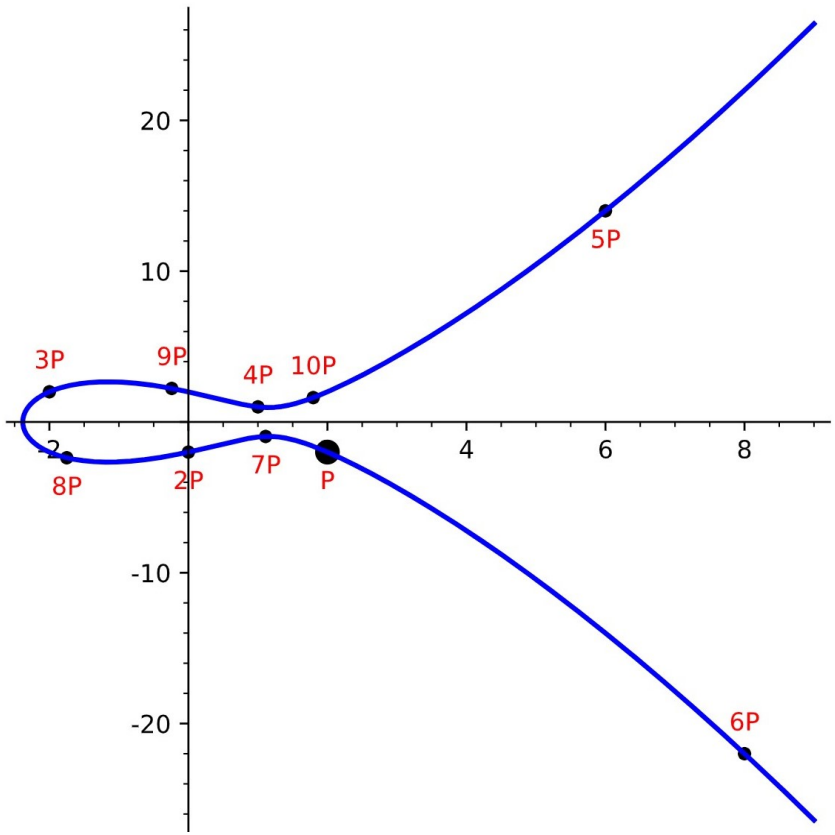

The curve $E : y^2 = x^3 - 4x + 4$ (`88.a1`) has trivial torsion subgroup and rank 1, with $E(\mathbb{Q}) = \langle (2, -2) \rangle \cong \mathbb{Z}$.

How about the rank?

- Computable?
- Classified?
- Parametrized in families?
- Statistically understood?

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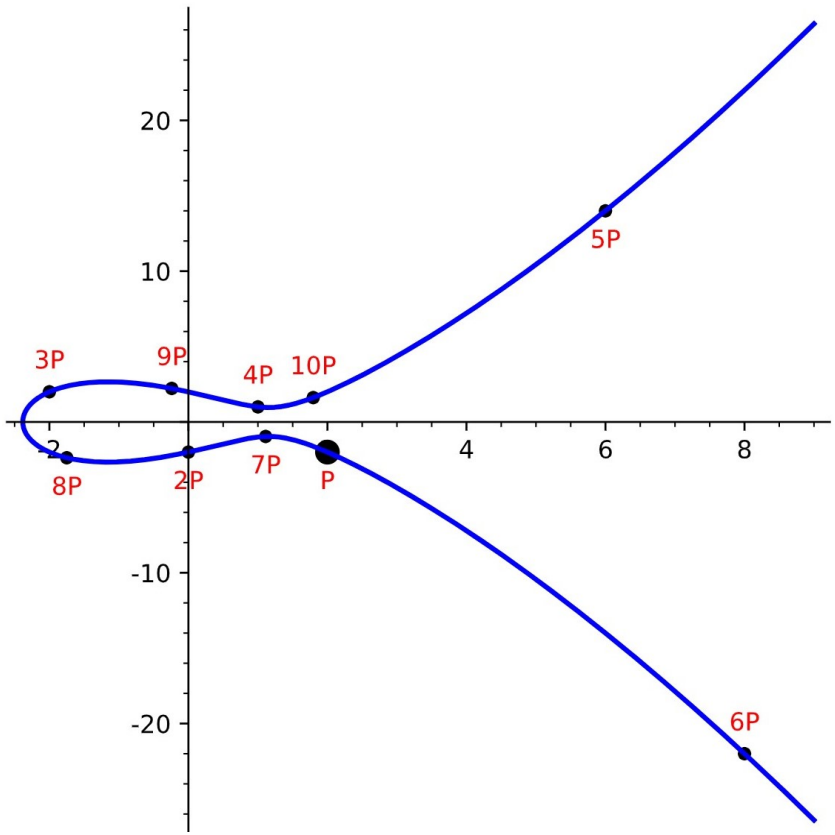

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How about the rank?

- ~~Computable?~~ Maybe
- ~~Classified?~~ No
- ~~Parametrized in families?~~ No
- ~~Statistically understood?~~ No

Is the Rank Computable?

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- **Analytically?** Yes*, if we assume B–S-D, the rank is the order of vanishing of the Hasse–Weil L-function $L(E, s)$ at $s = 1$.
  (* Computing values requires $\approx \sqrt{N_E}$ Fourier coefficients, and issues certifying zeroes numerically.)

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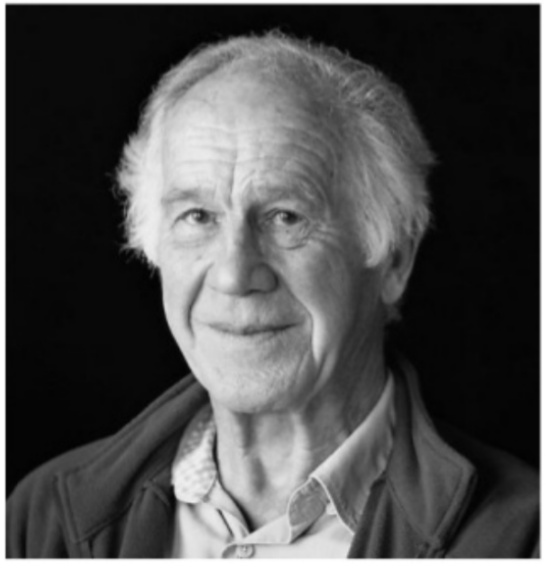
The **Birch and Swinnerton-Dyer conjecture** is wide open, with only some special cases (rank ≤ 1) known to be true.



Bryan Birch

Sir Peter Swinnerton-Dyer

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The **Birch and Swinnerton-Dyer conjecture** is wide open, with a one million dollar reward attached to it (it is one of the Millenium Problems proposed by the Clay Math Institute).

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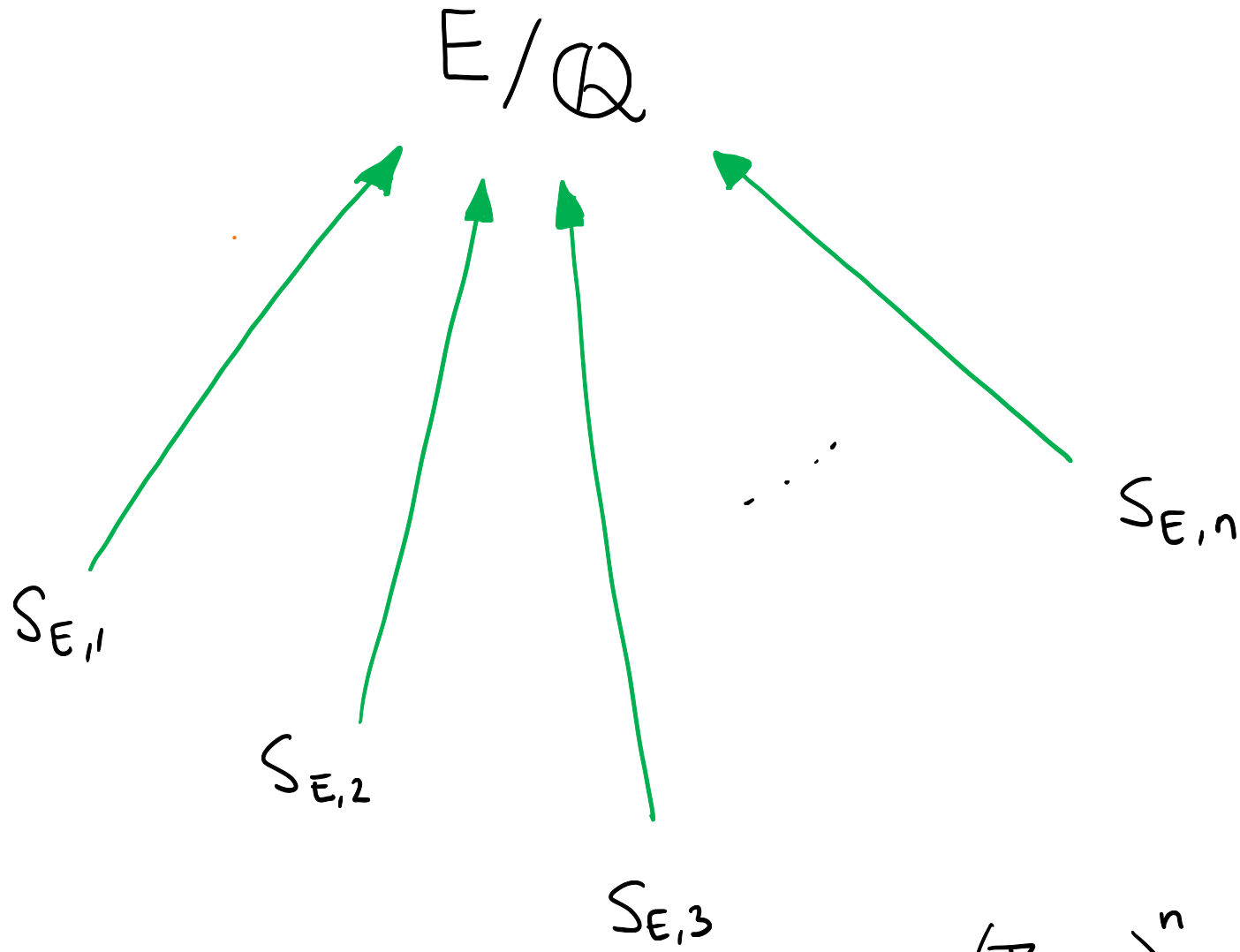
  The method of descent is based on the following exact sequence:

$$0 \longrightarrow E(\mathbb{Q})/p^n E(\mathbb{Q}) \longrightarrow \mathrm{Sel}_{p^n}(E/\mathbb{Q}) \longrightarrow Ш(E/\mathbb{Q})[p^n] \longrightarrow 0,$$

  where $\mathrm{Sel}_{p^n}(E/\mathbb{Q})$ is a finite, computable, cohomological group defined by finitely many local conditions.

2-Descent

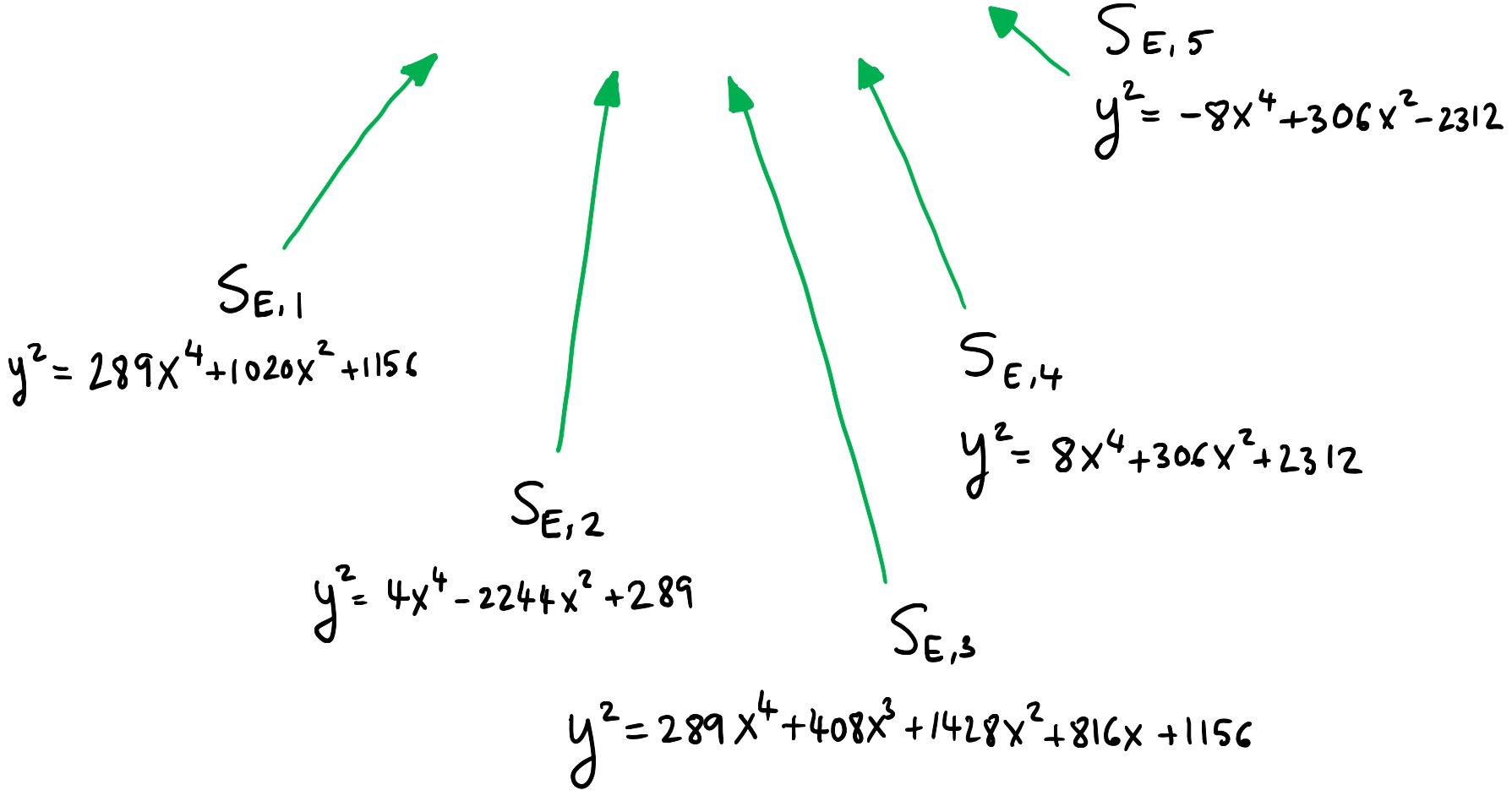


where $Sel_2(E/\mathbb{Q}) = \langle S_{E,1}, S_{E,2}, \ldots, S_{E,n} \rangle \cong \left(\mathbb{Z}/2\mathbb{Z}\right)^n$

2-Descent: Example

$E/\mathbb{Q}$

$$y^2 = x^3 - 105196x - 12970320$$



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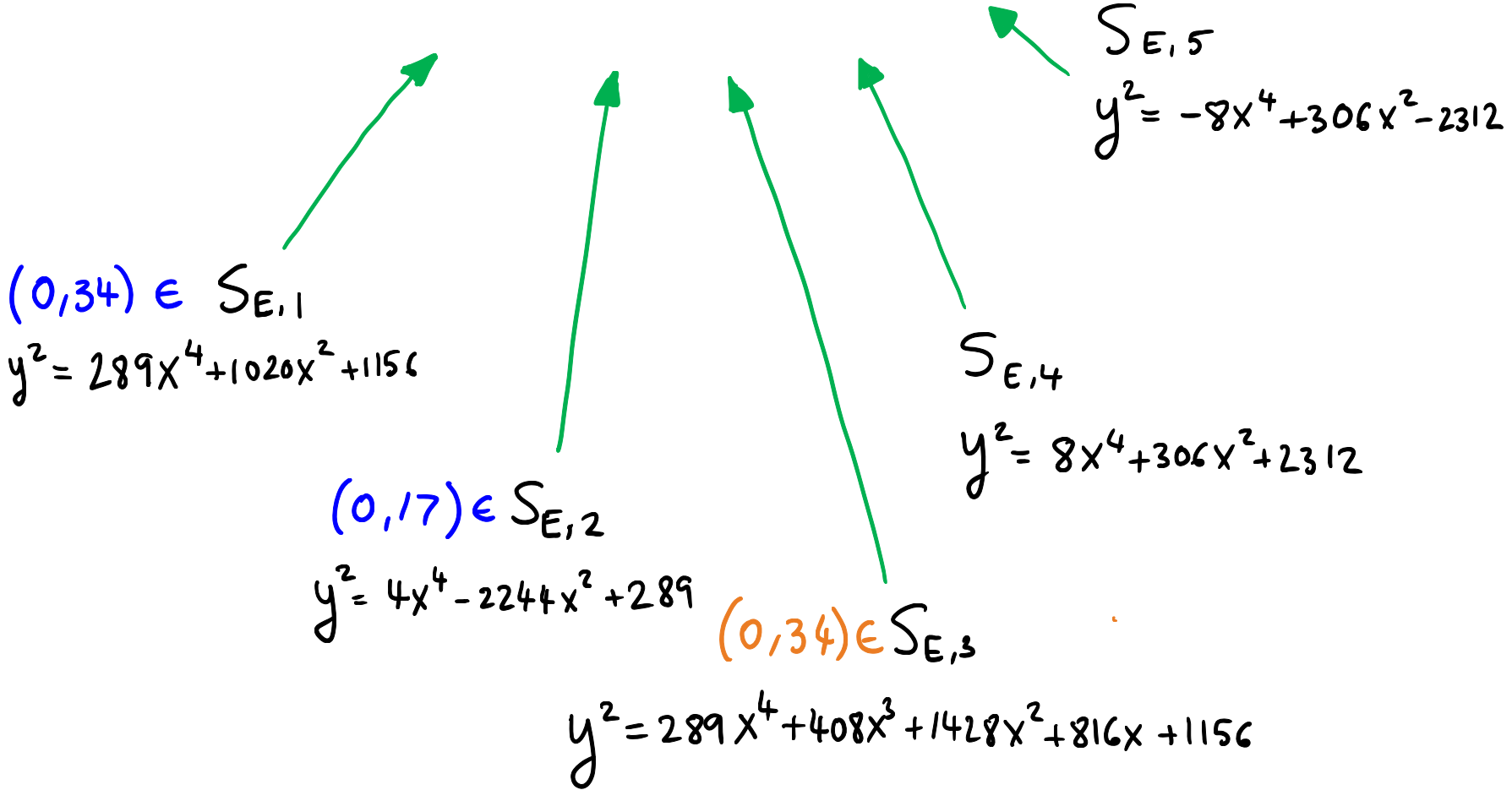

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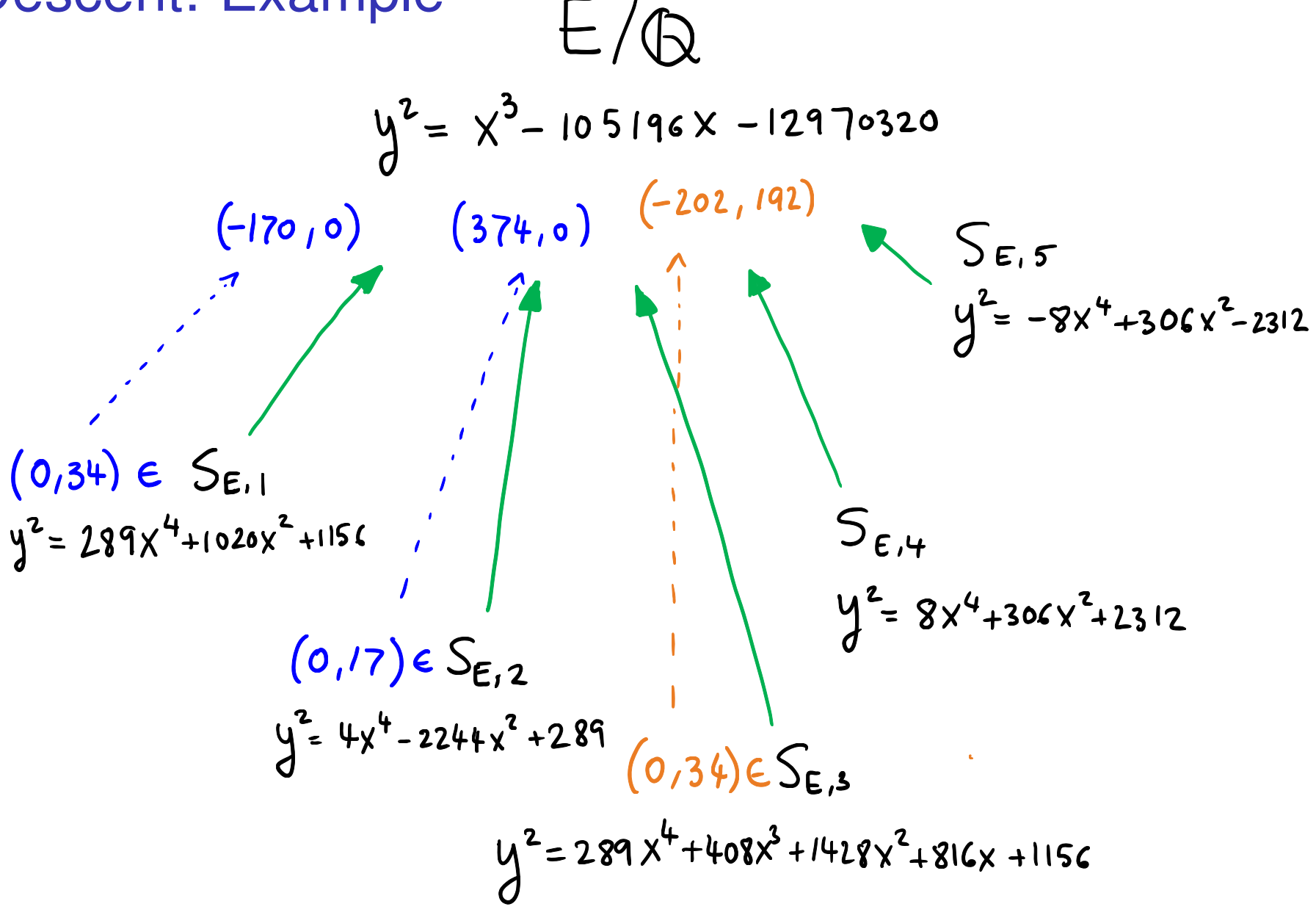

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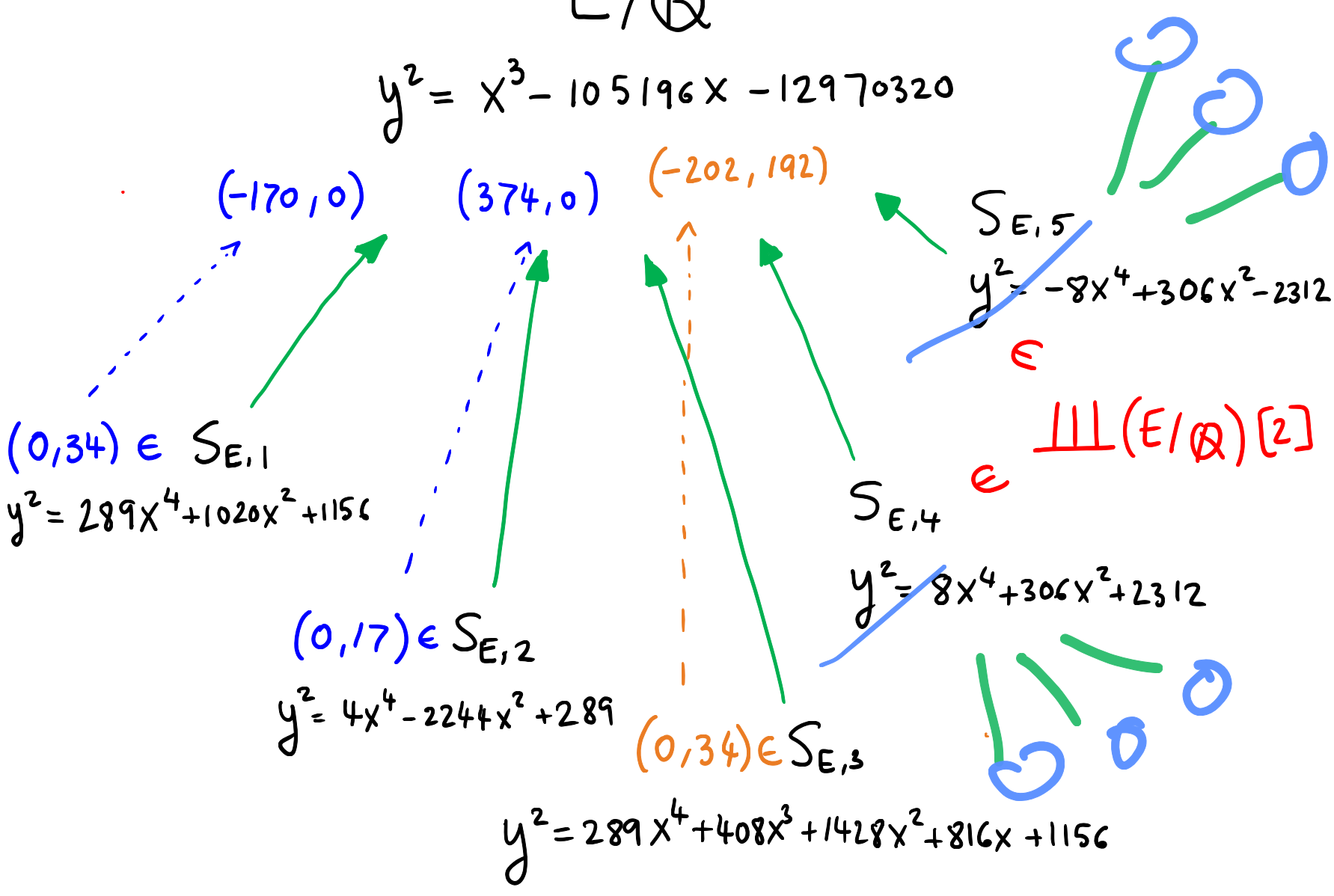

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$$y^2 = x^3 - 105196x - 12970320$$

$(-170, 0)$ $(374, 0)$ $(-202, 192)$

$S_{E,5}$

$y^2 = -8x^4 + 306x^2 - 2312$

$\in$

$Ш(E/\mathbb{Q})[2]$

$\in$

$(0, 34) \in S_{E,1}$

$y^2 = 289x^4 + 1020x^2 + 1156$

$S_{E,4}$

$y^2 = 8x^4 + 306x^2 + 2312$

$(0, 17) \in S_{E,2}$

$y^2 = 4x^4 - 2244x^2 + 289$

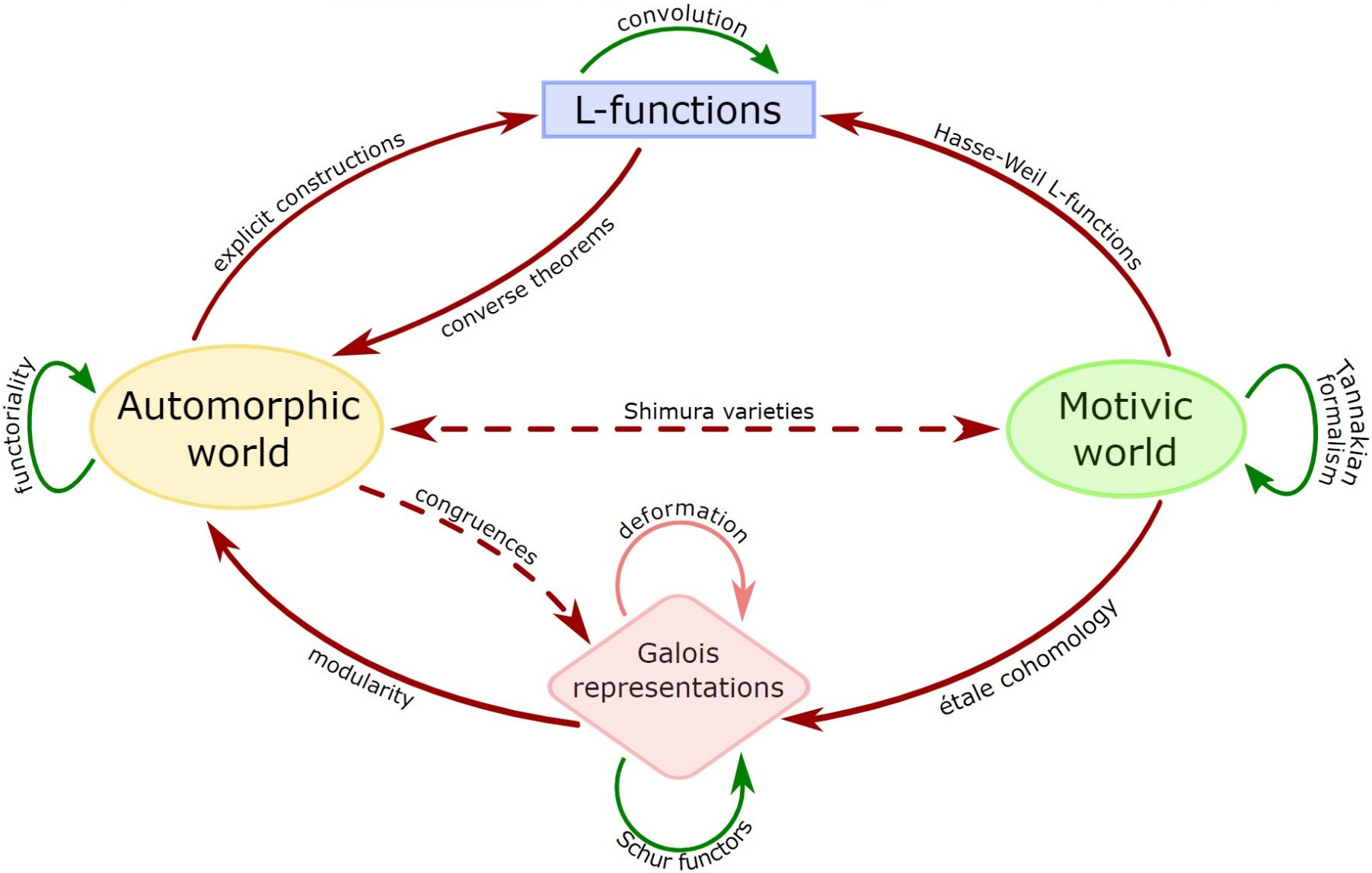
$(0, 34) \in S_{E,3}$

$y^2 = 289x^4 + 408x^3 + 1428x^2 + 816x + 1156$

$$Sel_2(E/\mathbb{Q}) \cong \left(\mathbb{Z}/2\mathbb{Z}\right)^2 \oplus \mathbb{Z}/2\mathbb{Z} \oplus \left(\mathbb{Z}/2\mathbb{Z}\right)^2$$

TORSION MW RANK ŠHA

The BIG Picture (the LMFDB universe)



THANK YOU

"*If by chance I have omitted anything*
more or less proper or necessary,
I beg forgiveness,
since there is no one who is without fault
and circumspect in all matters."

Leonardo "Bigollo" Pisano, *Liber Abaci*.